Linear conjunctive languages are closed under complement

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Technical report 2002-455

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January 2002

Abstract

Linear conjunctive grammars are conjunctive grammars in which the body of each conjunct contains no more than a single nonterminal symbol. They can at the same time be thought of as a special case of conjunctive grammars and as a generalization of linear context-free grammars.

While the problem of whether the complement of any conjunctive language can be denoted with a conjuctive grammar is still open and conjectured to have a negative answer, it turns out that the subfamily of linear conjunctive languages is in fact closed under complement and therefore under all set-theoretic operations.

1 Introduction

Conjunctive grammars, introduced in [1, 2], are context-free grammars augmented with an explicit set-theoretic intersection operation.

A grammar is defined as a quadruple $G = (\Sigma, N, P, S)$, where Σ and N are disjoint finite nonempty sets of terminal and nonterminal symbols respectively; P is a finite set of grammar rules, each of the form

$$A \to \alpha_1 \& \ldots \& \alpha_n \quad (A \in N,\ n \geqslant 1,\ \forall i\ \alpha_i \in (\Sigma \cup N)^*), \tag{1}$$

where the strings α_i are distinct, and their order is considered insignificant in the sense that there are no two rules in P that differ only in the order of these strings; $S \in N$ is a nonterminal designated as the start symbol. For each rule of the form (1) and for each i ($1 \leqslant i \leqslant n$), $A \to \alpha_i$ is called a conjunct. Let $conjuncts(G)$ denote the set of all conjuncts.

Terminal strings are derived by a conjunctive grammar by transforming strings over the alphabet $\Sigma \cup N\{$"(", "&", ")"$\}$ in the following way: nonterminals can be replaced with rule bodies enclosed in parentheses (e.g., by the rule (1) "A" can be replaced with "$(\alpha_1 \& \ldots \& \alpha_n)$"), and conjunction of several terminal strings in parentheses can be replaced with one such string (e.g., a substring "$(u\&u)$" can be replaced with "u"). The language generated by a conjunctive grammar is the set of all strings over Σ that can be derived from S.

Similarly to the linear context-free grammars, linear conjunctive grammars are those where the body of each conjunct contains no more than one nonterminal symbol, i.e. each string α_i in (1) is either in Σ^* or in $\Sigma^* \cdot N \cdot \Sigma^*$. There is no loss of generality in the stronger assumption that every rule is either of the form

$$A \to u_1 B_1 v_1 \& \ldots \& u_m B_m v_m \quad (u_i, v_i \in \Sigma^*,\ B_i \in N) \tag{2a}$$

or of the form

$$A \to w \quad (w \in \Sigma^*) \tag{2b}$$

It has been proved in [2] that every linear conjunctive grammar can be effectively transformed to an equivalent grammar in the so-called linear normal form, where each rule is of the form

$$A \to bB_1 \& \ldots \& bB_m \& C_1 c \& \ldots \& C_n c \quad (m + n \geqslant 1;\ A, B_i, C_j \in N;\ b, c \in \Sigma), \tag{3a}$$

$$A \to a \quad (A \in N, a \in \Sigma), \tag{3b}$$

$$S \to \epsilon, \quad \text{only if } \ S \text{ does not appear in right parts of rules} \tag{3c}$$

Both the family of linear conjunctive languages and the whole family of conjunctive languages are obviously closed under union and intersection, since for any two given conjunctive (linear conjunctive) grammars $G_i = (\Sigma, N_i, P_i, S_i)$ $(i = 1, 2)$ it is possible to construct the conjunctive (linear conjunctive) grammar $G = (\Sigma, N_1 \cup N_2 \cup \{S\}, P_1 \cup P_2 \cup P_3, S)$, where $P_3 = \{S \to S_1,\ S \to S_2\}$ for the case of union and $P_3 = \{S \to S_1 \& S_2\}$ for the case of intersection.

Concerning the closure under complement, it has been conjectured [2] that the whole family of conjunctive languages is not closed under this operation and that the co-context-free language $\{ww \mid w \in \Sigma^*\}$ is not conjunctive for any alphabet of cardinality 2 or more. This conjecture has neither been proved nor disproved, and it remains unknown whether the family of conjunctive languages is closed under complement.

In this paper we prove that for each linear conjunctive grammar G there exists a linear conjunctive grammar that generates the complement of $L(G)$, which means the closure of this language family under complement. The informal reasoning behind the construction is given in Section 2, and Section 3 describes an algorithmic grammar transformation that yields the grammar for the complement of the given language.

In Section 4 we prove that the constructed grammar indeed generates the complement of the language generated by the source grammar.

Finally, in Section 5 we apply this formal transformation to construct a linear conjunctive grammar for the complement of the language $\{wcw \mid w \in \{a, b\}^*\}$.

2 General idea

A string w is generated by some nonterminal A if and only if there is a rule

$$A \to s_{11} \dots s_{1n_1} \& \dots \& s_{m1} \dots s_{mn_m} \quad (s_{ij} \in \Sigma \cup N) \tag{4}$$

in the grammar, such that for every i $(1 \leqslant i \leqslant m)$ there is a factorization $w = u_1 \dots u_{n_i}$, where $u_j \in L(s_{ij})$ for all j $(1 \leqslant j \leqslant n_i)$.

By writing out a formal negation of this statement, we get that a string w is *not* generated by some nonterminal A if and only if for every rule of the form (4) there exists some i $(1 \leqslant i \leqslant m)$, such that for every factorization $w = u_1 \dots u_{n_i}$ there is j $(1 \leqslant j \leqslant n_i)$, for which $u_j \notin L(s_{ij})$.

While those universal and existential quantifiers in the latter statement that refer to rules and conjuncts could be implemented by the means of set-theoretic intersection and union (represented with conjunction of several strings in one rule and multiple rules for one nonterminal respectively), there is no obvious way to express within the formalism of conjunctive languages that some condition must hold *for every factorization*. That is why no method to construct a direct negation of a given conjunctive grammar is known, and it is conjectured [2] that the family is not closed under complement.

However, the situation changes if we restrict ourselves to linear conjunctive grammars. If a conjunct is of the form $A \to uBv$, where $u, v \in \Sigma^*$, then there is no more than one "meaningful" factorization of each string, because $w \in L(uBv)$ if and only if $w = uxv$ for some $x \in L(B)$; the same is true in respect to the conjuncts of the form $A \to u$.

In light of this singularity the difference between the existential and the universal quantifier on factorizations vanishes, and thus the aforementioned condition of a string's not being derivable from a nonterminal can be reformulated as follows: If a conjunctive grammar is linear, then a string w is *not* generated by some nonterminal A if and only if for every rule of the form (4) there exists a number i $(1 \leqslant i \leqslant m)$, a factorization $w = u_1 \dots u_{n_i}$, and a number j $(1 \leqslant j \leqslant n_i)$, such that $u_j \notin L(s_{ij})$.

This condition turns out to be expressible in the terms of linear conjunctive grammars. Let us develop an effective method to construct a linear conjunctive grammar for the complement of the language generated by a given linear conjunctive grammar.

3 Construction

Let $G = (\Sigma, N, P, S)$ be an arbitrary linear conjunctive grammar in the linear normal form.

We construct the following linear conjunctive grammar:

$$G' = (\Sigma, N_X \cup N_Y \cup N_Z \cup N_U \cup N_V \cup N_W, P', S') \tag{5a}$$

where

$$N_X = \{X_{\neg A} \mid A \in N\}, \tag{5b}$$
$$N_Y = \{Y_{\neg A \to \alpha_1 \& \ldots \& \alpha_m} \mid A \to \alpha_1 \& \ldots \& \alpha_m \in P\}, \tag{5c}$$
$$N_Z = \{Z_{\neg a} \mid a \in \Sigma\} \cup \{Z_{\neg \epsilon}\}, \tag{5d}$$
$$N_U = \{U_{\neg a \Sigma^+} \mid a \in \Sigma\}, \tag{5e}$$
$$N_V = \{V_{\neg \Sigma^+ a} \mid a \in \Sigma\}, \tag{5f}$$
$$N_W = \{W\}, \tag{5g}$$
$$S' = X_{\neg S} \tag{5h}$$

The nonterminals of the new grammar are meant to generate the following languages:

- For each nonterminal symbol $A \in N$ of the original grammar, the nonterminal $X_{\neg A}$ generates the complement of $L_G(A)$, i.e.

$$L_{G'}(X_{\neg A}) = \Sigma^* \setminus L_G(A) \tag{6}$$

- For each rule $A \to \alpha_1 \& \ldots \& \alpha_m \in P$, the nonterminal $Y_{\neg A \to \alpha_1 \& \ldots \& \alpha_m}$ generates those and only those strings that are not generated by this rule in the original grammar:

$$L_{G'}(Y_{\neg A \to \alpha_1 \& \ldots \& \alpha_m}) = \Sigma^* \setminus L_G((\alpha_1 \& \ldots \& \alpha_m)) \tag{7}$$

- For each terminal symbol $a \in \Sigma$, the nonterminal $Z_{\neg a}$ generates all strings but the string a. $Z_{\neg \epsilon}$ generates Σ^+.

- For each $a \in \Sigma$, the nonterminal $U_{\neg a \Sigma^+}$ generates all strings except those in $a \cdot \Sigma^+$, i.e. ϵ, all one-symbol strings and and all strings of the form bw ($w \in \Sigma^+$, $b \in \Sigma$, $b \neq a$).

- Similarly, $V_{\neg \Sigma^+ a}$ generates all strings except those that are end with a and are at least two symbols long.

- Finally, the nonterminal W generates Σ^*.

Now let us construct P', the set of rules of the new grammar:

- For each nonterminal $A \in N$ of the original grammar, if there are no rules for A in P, then P' contains the following single rule for $X_{\neg A}$:

$$X_{\neg A} \to W \tag{8a}$$

  Otherwise, if P contains some rules for A, let $\{A \to \mathcal{A}_1, \ldots A \to \mathcal{A}_m\}$ denote all these rules. Then the new grammar contains the following rule for $X_{\neg A}$:

$$X_{\neg A} \to Y_{\neg A \to \mathcal{A}_1} \& \ldots \& Y_{\neg A \to \mathcal{A}_m} \tag{8b}$$

- For each rule $A \to bB_1\&\ldots\&bB_m\&C_1c\&\ldots\&C_nc \in P$ $(m+n \geqslant 1)$ of the original grammar, the new grammar contains the following rules:

$$Y_{\neg A\to bB_1\&\ldots\&bB_m\&C_1c\&\ldots\&C_nc} \to U_{\neg b\Sigma^+} \quad (\text{if } m>0) \tag{9a}$$
$$Y_{\neg A\to bB_1\&\ldots\&bB_m\&C_1c\&\ldots\&C_nc} \to V_{\neg \Sigma^+ c} \quad (\text{if } n>0) \tag{9b}$$
$$Y_{\neg A\to bB_1\&\ldots\&bB_m\&C_1c\&\ldots\&C_nc} \to bX_{\neg B_i} \quad (\text{for all } i \in \{1,\ldots,m\}) \tag{9c}$$
$$Y_{\neg A\to bB_1\&\ldots\&bB_m\&C_1c\&\ldots\&C_nc} \to X_{\neg C_j}c \quad (\text{for all } j \in \{1,\ldots,n\}) \tag{9d}$$

- For each rule $A \to a \in P$ of the original grammar, the new grammar has the rule
$$Y_{\neg A\to a} \to Z_{\neg a} \tag{10}$$

- If the original grammar contains the rule $S \to \epsilon$, then there is a rule
$$Y_{\neg S\to \epsilon} \to Z_{\neg \epsilon} \tag{11}$$

- The languages generated by the nonterminals from $N_Z \cup N_U \cup N_V \cup N_W$ are all regular and thus linear conjunctive; writing the rules for them does not pose any difficulty.

4 Proof

Let us prove the correctness of our construction.

Lemma 1. *Let $G = (\Sigma, N, P, S)$ be a linear conjunctive grammar in the linear normal form. Let G' be a grammar constructed from G using the method given in Section 3.*

Then for every $n \geqslant 0$ and for every nonterminal $A \in N$, $L_G(A) = \Sigma^* \setminus L_{G'}(X_{\neg A}) \pmod{\Sigma^{\leqslant n}}$.

Proof. Induction on n.

Basis $n = 1$ Let $w \in \{\epsilon\} \cup \Sigma$ and let $A \in N$. Due to G's being in the linear normal form, $w \in L_G(A)$ if and only if $A \to w \in P$ (which is either $S \to \epsilon$ or $A \to a$).

Let $A \to w \in P$. Then $Y_{\neg A\to w}$ does not generate w, because the only rule for $Y_{\neg A\to w}$ is $Y_{\neg A\to w} \to Z_{\neg w}$. Therefore, the only rule for $X_{\neg A}$ contains a conjunct $(X_{\neg A} \to Y_{\neg S\to\epsilon} \in \mathit{conjuncts}(P'))$ that does not generate the string w, and hence $\epsilon \notin L_{G'}(X_{\neg A})$.

Now let $A \to w \notin P$. This means that all rules for A are either of the form

$$A \to bB_1\&\ldots\&bB_m\&C_1c\&\ldots\&C_nc \in P \quad (m+n \geqslant 1) \tag{12a}$$

or of the form

$$A \to a \quad (a \in \Sigma,\ a \neq w) \tag{12b}$$

For each rule of the form (12a) the corresponding nonterminal $Y_{\neg A \to bB_1\&\ldots\&bB_m\&C_1c\&\ldots\&C_nc}$ has at least one of the rules (9a) and (9b) and hence generates ϵ. The same holds in respect to each rule of the form (12b), because the nonterminal $Y_{\neg A \to a}$ has the rule (10), which can generate any string except a, and hence the string w. Since every conjunct of the only rule for nonterminal $X_{\neg A}$ generates w, so does the nonterminal $X_{\neg A}$.

Induction step Assume $L_G(A) = \Sigma^* \setminus L_{G'}(X_{\neg A}) \pmod{\Sigma^{\leqslant n-1}}$ for some $n \geqslant 2$ and consider an arbitrary nonterminal $A \in N$ and an arbitrary string $w \in \Sigma^n$. Denote $w = bxc$, where $b, c \in \Sigma$ and $x \in \Sigma^{n-2}$.

$w \in L_G(A)$ if and only if there exists some rule

$$A \to bB_1\& \ldots \&bB_m\&C_1c\& \ldots \&C_nc \in P \quad (m + n \geqslant 1) \tag{13}$$

such that there exist derivations

$$B_i \stackrel{G}{\Longrightarrow} \ldots \stackrel{G}{\Longrightarrow} xc \quad (\text{for all } i\text{: } 1 \leqslant i \leqslant m) \tag{14a}$$

$$C_j \stackrel{G}{\Longrightarrow} \ldots \stackrel{G}{\Longrightarrow} bx \quad (\text{for all } j\text{: } 1 \leqslant j \leqslant n) \tag{14b}$$

By induction hypothesis, (14) holds if and only if for some rule (13)

$$xc \notin L_{G'}(X_{\neg B_i}) \quad (\text{for all } i\text{: } 1 \leqslant i \leqslant m) \tag{15a}$$

$$bx \notin L_{G'}(X_{\neg C_j}) \quad (\text{for all } j\text{: } 1 \leqslant j \leqslant n) \tag{15b}$$

Now let us see that $Y_{\neg A \to bB_1\&\ldots\&bB_m\&C_1c\&\ldots\&C_nc}$ does not generate w if and only if (15) is the case. Indeed, if we assume (15), then none of the rules (9) derive w: (9a) cannot derive bxc because it starts with b and is at least two symbols long, (9b) does not derive it because it starts with c, and the rules of the form (9c) and (9d) are of no use due to (15).

If (15) is untrue, then xc is in $L_{G'}(X_{\neg B_i})$ for some i (or bx is in $L_{G'}(X_{\neg C_j})$ for some i), and hence one of the rules

$$Y_{\neg A \to bB_1\&\ldots\&bB_m\&C_1c\&\ldots\&C_nc} \to bX_{\neg B_i} \tag{16a}$$

$$Y_{\neg A \to bB_1\&\ldots\&bB_m\&C_1c\&\ldots\&C_nc} \to X_{\neg C_j}c \tag{16b}$$

derives w.

Putting together the results we have established so far, $w \in L_G(A)$ if and only if there exists a rule (13) of the original grammar, such that

$$w \notin L_{G'}(Y_{\neg A \to bB_1\&\ldots\&bB_m\&C_1c\&\ldots\&C_nc}) \tag{17}$$

But the existence of such a rule implies the existence of a conjunct of the rule (8b) that does not derive w. Since (8b) is the only rule for $X_{\neg A}$, we conclude that $w \in L_G(A)$ if and only if $w \notin L_{G'}(X_{\neg A})$, which, due to the arbitrariness of the choice of w, proves the induction step.

This completes the proof. $\square$

Corollary 1. *The languages $L_G(A)$ and $\Sigma^* \setminus L_{G'}(X_{\neg A})$ coincide.*

Corollary 2. $L(G) = \Sigma^* \setminus L(G')$.

Theorem 1. *The family of linear conjunctive languages is closed under complement.*

5 Example

Consider the following linear conjunctive grammar for the language $L = \{wcw \mid w \in \{a,b\}^*\}$ [1, 2]:

$$\begin{aligned}
&S \to C\&K\\
&C \to aCa \mid aCb \mid bCa \mid bCb \mid c\\
&K \to aA\&aK \mid bB\&bK \mid cE\\
&A \to aAa \mid aAb \mid bAa \mid bAb \mid cRa\\
&B \to aBa \mid aBb \mid bBa \mid bBb \mid cRb\\
&R \to aR \mid bR \mid \epsilon
\end{aligned}$$

After converting it to linear normal form using the method given in [2] we obtain the following equivalent grammar:

$$\begin{aligned}
&S \to Da\&aA\&aK \mid Db\&aA\&aK \mid Ea\&bB\&bK \mid Eb\&bB\&bK \mid c\\
&C \to Da \mid Db \mid Ea \mid Eb \mid c\\
&D \to aC\\
&E \to bC\\
&K \to aA\&aK \mid bB\&bK \mid cR \mid c\\
&A \to aI \mid aJ \mid bI \mid bJ \mid Fa\\
&I \to Aa\\
&J \to Ab\\
&B \to aM \mid aN \mid bM \mid bN \mid Fb\\
&M \to Ba\\
&N \to Bb\\
&R \to aR \mid bR \mid a \mid b\\
&F \to cR \mid c
\end{aligned}$$

The resulting grammar for the complement of L contains the following rules:

$$\begin{aligned}
&X_{\neg S} \to Y_{\neg S \to Da\&aA\&aK} \& Y_{\neg S \to Db\&aA\&aK} \& Y_{\neg S \to Ea\&bB\&bK} \&\\
&\qquad\quad Y_{\neg S \to Eb\&bB\&bK} \& Y_{\neg S \to c}\\
&Y_{\neg S \to Da\&aA\&aK} \to U_{\neg a\Sigma^+} \mid V_{\neg \Sigma^+ a} \mid X_{\neg D}a \mid aX_{\neg A} \mid aX_{\neg K}\\
&Y_{\neg S \to Db\&aA\&aK} \to U_{\neg a\Sigma^+} \mid V_{\neg \Sigma^+ b} \mid X_{\neg D}b \mid aX_{\neg A} \mid aX_{\neg K}\\
&Y_{\neg S \to Ea\&bB\&bK} \to U_{\neg b\Sigma^+} \mid V_{\neg \Sigma^+ a} \mid X_{\neg E}a \mid bX_{\neg B} \mid bX_{\neg K}\\
&Y_{\neg S \to Eb\&bB\&bK} \to U_{\neg b\Sigma^+} \mid V_{\neg \Sigma^+ b} \mid X_{\neg E}b \mid bX_{\neg B} \mid bX_{\neg K}\\
&Y_{\neg S \to c} \to Z_{\neg c}
\end{aligned}$$

$X_{\neg C} \to Y_{\neg C\to Da} \& Y_{\neg C\to Db} \& Y_{\neg C\to Ea} \& Y_{\neg C\to Eb} \& Y_{\neg C\to c}$
$Y_{\neg C\to Da} \to V_{\neg\Sigma^+ a} \mid X_{\neg D}a$
$Y_{\neg C\to Db} \to V_{\neg\Sigma^+ b} \mid X_{\neg D}b$
$Y_{\neg C\to Ea} \to V_{\neg\Sigma^+ a} \mid X_{\neg E}a$
$Y_{\neg C\to Eb} \to V_{\neg\Sigma^+ b} \mid X_{\neg E}b$
$Y_{\neg C\to c} \to Z_{\neg c}$

$X_{\neg D} \to Y_{\neg D\to aC}$
$Y_{\neg D\to aC} \to U_{\neg a\Sigma^+} \mid aX_{\neg C}$

$X_{\neg E} \to Y_{\neg E\to bC}$
$Y_{\neg E\to bC} \to U_{\neg b\Sigma^+} \mid bX_{\neg C}$

$X_{\neg K} \to Y_{\neg K\to aA\&aK} \& Y_{K\to bB\&bK} \& Y_{K\to cR} \& Y_{K\to c}$
$Y_{\neg K\to aA\&aK} \to U_{\neg a\Sigma^+} \mid aX_{\neg A} \mid aX_{\neg K}$
$Y_{\neg K\to bA\&bK} \to U_{\neg b\Sigma^+} \mid bX_{\neg A} \mid bX_{\neg K}$
$Y_{\neg K\to cR} \to U_{\neg c\Sigma^+} \mid cX_{\neg R}$
$Y_{\neg K\to c} \to Z_{\neg c}$

$X_{\neg A} \to Y_{\neg A\to aI} \& Y_{\neg A\to aJ} \& Y_{\neg A\to bI} \& Y_{\neg A\to bJ} \& Y_{\neg A\to Fa}$
$Y_{\neg A\to aI} \to U_{\neg a\Sigma^+} \mid aX_{\neg I}$
$Y_{\neg A\to aJ} \to U_{\neg a\Sigma^+} \mid aX_{\neg J}$
$Y_{\neg A\to bI} \to U_{\neg b\Sigma^+} \mid bX_{\neg I}$
$Y_{\neg A\to bJ} \to U_{\neg b\Sigma^+} \mid bX_{\neg J}$
$Y_{\neg A\to Fa} \to V_{\neg\Sigma^+ a} \mid X_{\neg F}a$

$X_{\neg I} \to Y_{\neg I\to Aa}$
$Y_{\neg I\to Aa} \to V_{\neg\Sigma^+ a} \mid X_{\neg A}a$

$X_{\neg J} \to Y_{\neg J\to Ab}$
$Y_{\neg J\to Ab} \to V_{\neg\Sigma^+ b} \mid X_{\neg A}b$

$X_{\neg B} \to Y_{\neg B\to aM} \& Y_{\neg B\to aN} \& Y_{\neg B\to bM} \& Y_{\neg B\to bN} \& Y_{\neg B\to Fb}$
$Y_{\neg B\to aM} \to U_{\neg a\Sigma^+} \mid aX_{\neg M}$
$Y_{\neg B\to aN} \to U_{\neg a\Sigma^+} \mid aX_{\neg N}$
$Y_{\neg B\to bM} \to U_{\neg b\Sigma^+} \mid bX_{\neg M}$
$Y_{\neg B\to bN} \to U_{\neg b\Sigma^+} \mid bX_{\neg N}$
$Y_{\neg B\to Fb} \to V_{\neg\Sigma^+ b} \mid X_{\neg F}b$

$X_{\neg M} \to Y_{\neg M\to Ba}$
$Y_{\neg M\to Ba} \to V_{\neg\Sigma^+ a} \mid X_{\neg B}a$

$X_{\neg N} \to Y_{\neg N\to Bb}$
$Y_{\neg N\to Bb} \to V_{\neg\Sigma^+ b} \mid X_{\neg B}b$

$X_{\neg R} \to Y_{\neg R\to aR} \& Y_{\neg R\to bR} \& Y_{\neg R\to a} \& Y_{\neg R\to b}$
$Y_{\neg R\to aR} \to U_{\neg a\Sigma^+} \mid aX_{\neg R}$
$Y_{\neg R\to bR} \to U_{\neg b\Sigma^+} \mid bX_{\neg R}$
$Y_{\neg R\to a} \to Z_{\neg a}$
$Y_{\neg R\to b} \to Z_{\neg b}$

$X_{\neg F} \to Y_{\neg F\to cR} \& Y_{\neg F\to c}$
$Y_{\neg F\to cR} \to U_{\neg c\Sigma^+} \mid cX_{\neg R}$
$Y_{\neg F\to c} \to Z_{\neg c}$

$$
\begin{aligned}
&Z_{\neg a} \to \epsilon \mid bW \mid cW \mid aaW \mid abW \mid acW\\
&Z_{\neg b} \to \epsilon \mid aW \mid cW \mid baW \mid bbW \mid bcW\\
&Z_{\neg c} \to \epsilon \mid aW \mid bW \mid caW \mid cbW \mid ccW\\
&Z_{\neg \epsilon} \to aW \mid bW \mid cW
\end{aligned}
$$

$$
\begin{aligned}
&U_{\neg a\Sigma^+} \to \epsilon \mid a \mid bW \mid cW\\
&U_{\neg b\Sigma^+} \to \epsilon \mid aW \mid b \mid cW\\
&U_{\neg c\Sigma^+} \to \epsilon \mid aW \mid bW \mid c
\end{aligned}
$$

$$
\begin{aligned}
&V_{\neg \Sigma^+ a} \to \epsilon \mid a \mid Wb \mid Wc\\
&V_{\neg \Sigma^+ b} \to \epsilon \mid Wa \mid b \mid Wc\\
&V_{\neg \Sigma^+ c} \to \epsilon \mid Wa \mid Wb \mid c
\end{aligned}
$$

$$
W \to aW \mid bW \mid cW \mid \epsilon
$$

6 Conclusion

We have proved that the family of linear conjunctive languages is closed under complement and hence, due to its obvious closure under union and intersection, is closed under all set-theoretic operations. In all cases the construction is effective in the sense it can be done by an algorithm.

The given method essentially uses the linearity of the grammar, and it seems very unlikely that it could be applied to the conjunctive grammars of general form. Thus the conjecture that the whole family of conjunctive languages is not closed under complement [2] remains in effect.

References

[1] A. Okhotin, "Conjunctive grammars", in *Pre-proceedings of DCAGRS 2000*, Dept. of Computer Science, University of Western Ontario, London, Ontario, Canada. Report No. 555 (2000).

[2] A. Okhotin, "Conjunctive grammars", *Journal of Automata, Languages and Combinatorics*, 6:4 (2001), to appear.