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AN APPLICATION OF PARALLEL COMPUTATION TO DYNAMICAL SYSTEMS*

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Abstract

An RLC circuit, certain parameters of which are measured sequentially, that is, one after the other, undergoes significant perturbations that affect its dynamical behavior. By contrast, these perturbations are usually eliminated when the measurements are performed in parallel, that is, when the parameters are measured simultaneously. This result confirms the existence of physical systems with the property that certain operations on them can be performed successfully in parallel but not sequentially.

Keywords: dynamical system, parallel computation, RLC-circuit, measurement, perturbation, damping, oscillation.

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1 Introduction

Parallel computation is a form of information processing whereby n processors, $n > 1$, cooperate to solve a computational problem by working on it simultaneously. The expectation is that this approach speeds up computations that would otherwise require an inordinate amount of time if performed sequentially (that is, using one processor). Over the last twenty five years, considerable progress has been achieved to fulfill the promise of parallel computation. Results, both in theory and in practice, were obtained to demonstrate that significant improvements are possible, not only in the *speed* with which a solution is obtained, but also in the *quality* of the solution itself.

In the aforementioned results, the level of improvement achieved through parallel computation varied over a wide range depending on the problem being solved, from sublinear, to linear, or even superlinear in the number of processors used on the parallel computer. Furthermore, those results were obtained within conventional paradigms (such as, for example, when all the data required by a computation are available at the outset), as well as unconventional paradigms (such as, for example, when the data arrive in real time and the results must be delivered by a certain strict deadline). For surveys of these results, see [1, 2, 4, 8].

An important characteristic of traditional analyses of parallel computation is that the conditions governing the computational environment are, in a fashion, fully determined by the human in charge and the model of computation used. For example, in a real-time computation, if it is deemed that the arrival rate of the data is too high, it is possible for the people responsible for the computation to slow down the arrival rate, or to extend the deadline by which a solution is to be delivered, or to use a faster computer, and so on.

A radical departure from this paradigm was taken recently. In [3], the focus is on computational environments in which a computation can succeed if and only if it is performed in parallel. In these environments, it is the laws of nature that prevail, rather than human-imposed computational circumstances or conditions on the computation. Specifically, it is shown that the principles governing such fields as physics, chemistry, and biology, are responsible for causing the inevitable failure of any sequential approach to solving the problem at hand, while at the same time allowing a parallel approach to succeed. A typical example of such principles is the uncertainty involved in measuring several related parameters of a physical system. Another principle expresses the way in which the components of a system in equilibrium react when subjected to outside stress.

In this paper we exhibit one such environment, namely, dynamical systems. In general, a *system* is a collection of elements that interact with one another. The system is characterized by a number of variables among which relationships of cause and effect hold. In particular, the system receives a number of inputs and produces a number of outputs based on these inputs. In a *static* system, the current values of the outputs depend only on the instantaneous values of the present inputs. If, on the other hand, the system has memory such that current outputs are based on present as well as past inputs, it is said to be a *dynamical* system. Here, variables are time-dependent. Excitations and responses vary with time. Moreover, the derivatives of variables with respect to time at any moment depend on the values of these variables at that moment [6]. Examples of dynamical systems include electrical systems, mechanical systems, thermal systems, fluid systems, and so on.

For the purposes of this paper, we have selected a very simple dynamical system in order to convey our point in the clearest possible way. A resistance-inductance-capacitance (RLC) circuit is used to demonstrate the thesis advanced in [3]. Specifically, we show that an RLC circuit, certain parameters of which are measured sequentially, that is, one after the other, undergoes significant perturbations that affect its dynamical behavior. By contrast, these perturbations are usually eliminated when the measurements are performed in parallel, that is, when the parameters are measured simultaneously. This result confirms the existence of physical systems with the property that certain operations on them can be performed successfully in parallel but not sequentially.

2 An RLC Circuit

Consider the circuit in Fig. 1 in which

E	is the voltage provided by a source (measured in volts)
I	is the current flowing in the circuit (measured in amperes)
R	is a resistor (offering a resistance measured in ohms)
L	is an inductor (offering an inductance measured in henries)
C	is a capacitor (offering a capacitance measured in farads)
U_C	is the voltage across C.

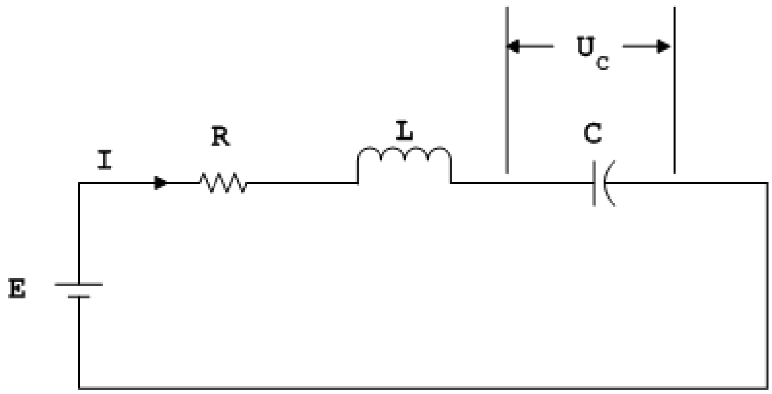

Figure 1: RLC circuit.

In this RLC circuit, the following relations hold:

$$L\frac{dI}{dt} = E - IR - U_C, \qquad C\frac{dU_C}{dt} = I. \tag{1}$$

In order to solve equation (1), we need initial conditions, or the values of I and U_C. Suppose that we choose to measure I and U_C. However, the act of measuring these two quantities may induce perturbations on the system as we now show.

Let us assume that the amperemeter and voltmeter used to measure I and U_C, respectively, have resistances R_1 and R_2, as shown in Fig. 2.

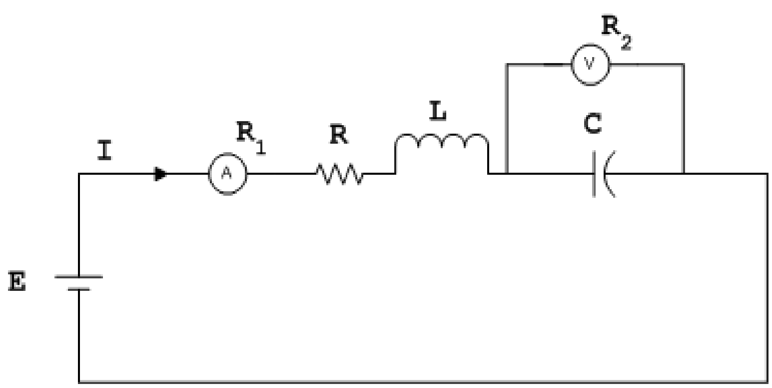

Figure 2: Measuring I and U_C.

The equations describing the system now become

$$L\frac{dI}{dt} = E - I(R + R_1) - U_C, \qquad C\frac{dU_C}{dt} = I - \frac{U_C}{R_2}. \tag{2}$$

In general, it is reasonable to assume that R_1 is very small and R_2 very large. Here, we focus on comparing the perturbations affecting the system when sequential and parallel measurement schemes are used.

For convenience, we transform equation (2) into standard homogeneous (ordinary differential equation) form, obtaining:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 + \delta a_1 & a_2 \\ a_3 & a_4 + \delta a_4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \tag{3}$$

where

$$x = I - I_0 \ , \ y = U_C - U_0 \ ,$$

$$I_0 = \frac{U_0}{R_2} \ , \ U_0 = \frac{ER_2}{(R + R_1 + R_2)} \ ,$$

$$a_1 = -\frac{R}{L} \ , \ \delta a_1 = -\frac{R_1}{L} \ ,$$

$$a_2 = -\frac{1}{L} \ , \ a_3 = \frac{1}{C} \ ,$$

$$a_4 = 0 \ \text{ and } \ \delta a_4 = -\frac{1}{R_2 C}.$$

In other words, the perturbations are on the diagonal elements of the parameter matrix

$$\begin{pmatrix} a_1 + \delta a_1 & a_2 \\ a_3 & a_4 + \delta a_4 \end{pmatrix}.$$

A general solution of equation (3) can be written as:

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \vec{v_1} e^{\lambda_1 t} + c_2 \vec{v_2} e^{\lambda_2 t} \tag{4}$$

where λ_1 and λ_2 are the eigenvalues of the parameter matrix, $\vec{v_1}$ and $\vec{v_2}$ are the corresponding eigenvectors, and c_1 and c_2 are coefficients which can be determined by the initial values of x and y. Equation (4) shows that the dynamical behavior of the system described by equation (3) is determined by the eigenvalues of the parameter matrix. Indeed, let the real and imaginary parts of eigenvalue λ_i be denoted by $\mathrm{Re}(\lambda_i)$ and $\mathrm{Im}(\lambda_i)$, respectively, for $i = 1, 2$. The quantities $\mathrm{Re}(\lambda_1)$ and $\mathrm{Re}(\lambda_2)$ determine whether or not the system is experiencing damping. Similarly, $\mathrm{Im}(\lambda_1)$ and $\mathrm{Im}(\lambda_2)$ represent the frequencies of the system oscillations. For example, if both $\mathrm{Re}(\lambda_1)$ and $\mathrm{Re}(\lambda_2)$ are less than zero, and one of $\mathrm{Im}(\lambda_1)$ and $\mathrm{Im}(\lambda_2)$ is not equal to zero, the system is in oscillation with damping.

The eigenvalues λ_1 and λ_2 are obtained from the parameter matrix in equation (3) as:

$$\lambda_{1,2} = \frac{1}{2}(a_1 + \delta a_1 + \delta a_4 \pm \sqrt{(a_1 + \delta a_1 - \delta a_4)^2 + 4a_2a_3}). \qquad (5)$$

When $a_1^2 + 4a_2a_3 \neq 0$, $|\delta a_1| \ll |\frac{a_1^2+4a_2a_3}{2a_1}|$, and $|\delta a_4| \ll |\frac{a_1^2+4a_2a_3}{2a_1}|$, equation (5) can be expanded into a Taylor series up to the first order of $|\delta a_1|$ and $|\delta a_4|$. Thus,

$$\begin{aligned}
\lambda_{1,2} &\simeq \lambda_{1,2}|_{\delta a_1=0,\delta a_4=0} + \frac{\partial \lambda_{1,2}}{\partial \delta a_1}|_{\delta a_1=0}\ \delta a_1 + \frac{\partial \lambda_{1,2}}{\partial \delta a_4}|_{\delta a_4=0}\ \delta a_4 \\
&= \lambda_{1,2}^{(0)} + \lambda_{1,2}^{(1)} + \lambda_{1,2}^{(2)}
\end{aligned}$$

where

$$\lambda_{1,2}^{(0)} = \frac{1}{2}\left(a_1 \pm \sqrt{a_1^2 + 4a_2a_3}\right),$$

$$\lambda_{1,2}^{(1)} = \frac{1}{2}\left(1 \pm \frac{a_1}{\sqrt{a_1^2 + 4a_2a_3}}\right)\delta a_1,$$

and

$$\lambda_{1,2}^{(2)} = \frac{1}{2}\left(1 \mp \frac{a_1}{\sqrt{a_1^2 + 4a_2a_3}}\right)\delta a_4.$$

If I and U_C are measured *sequentially*, that is, one after the other, the perturbations on the eigenvalues caused by the measurements are $\lambda_{1,2}^{(1)}$ and

$\lambda_{1,2}^{(2)}$, respectively. On the other hand, if I and U_C are measured *in parallel*, that is, at the same time, the perturbation is

$$\lambda_{1,2}^{(1)} + \lambda_{1,2}^{(2)} = \frac{1}{2}(\delta a_1 + \delta a_4) + \frac{1}{2}\frac{a_1}{\sqrt{a_1^2 + 4a_2a_3}}(\delta a_1 - \delta a_4). \tag{6}$$

Now recall that for the RLC circuit

$$a_1 = -\frac{R}{L},$$

$$a_1^2 + 4a_2a_3 = \frac{R^2}{L^2} - \frac{4}{LC},$$

and

$$\delta a_1 - \delta a_4 = -\frac{R_1}{L} + \frac{1}{R_2C}.$$

We now analyze the perturbations on the eigenvalues in the following cases:

Case 1 : $a_1^2 + 4a_2a_3 > 0$. This implies that

$$\frac{a_1}{\sqrt{a_1^2 + 4a_2a_3}} < -1.$$

Therefore, the term $\frac{a_1}{\sqrt{a_1^2+4a_2a_3}}$ plays a more important role than the constant term 1 in $\lambda_{1,2}^{(1)}$ and $\lambda_{1,2}^{(2)}$, especially when it is much smaller than -1. In this case, measuring one of I or U_C before the other may cause big perturbations on $\lambda_{1,2}$. However, the perturbations may be decreased if a simultaneous measurement scheme is adopted. This is because

$$|\delta a_1 - \delta a_4| = |-\frac{R_1}{L} + \frac{1}{R_2C}| < |\frac{R_1}{L}| + |\frac{1}{R_2C}| = |\delta a_1| + |\delta a_4|.$$

In particular, if

$$\frac{R_1}{L} = \frac{1}{R_2C},$$

the perturbation $\lambda_{1,2}^{(1)} + \lambda_{1,2}^{(2)}$ is such that

$$|\lambda_{1,2}^{(1)} + \lambda_{1,2}^{(2)}| = \frac{R_1}{L},$$

which is much smaller than $|\lambda_{1,2}^{(1)}|$ and $|\lambda_{1,2}^{(2)}|$.

Case 2 : $a_1^2 + 4a_2a_3 < 0$. Here, the term

$$\frac{a_1}{\sqrt{a_1^2 + 4a_2a_3}}$$

is imaginary. In this case, this term affects the frequency of the system oscillations. In particular, when $-a_1^2 - 4a_2a_3 \to 0$ and a_1 keeps limited, the frequency of the original system, namely,

$$\sqrt{-(a_1^2 + 4a_2a_3)}$$

is low but the perturbations on the frequency from a sequential measurement could be big, namely,

$$\frac{a_1}{\sqrt{-(a_1^2 + 4a_2a_3)}}\delta a_1, \quad \text{or} \quad \frac{a_1}{\sqrt{-(a_1^2 + 4a_2a_3)}}\delta a_4.$$

A simultaneous measurement scheme, however, may reduce the perturbations, especially when $\delta a_1 - \delta a_4$ is very small.

Finally, when $a_1^2 + 4a_2a_3 = 0$, we have from equation (5)

$$\lambda_{1,2} \simeq \frac{1}{2}\left(a_1 \pm \sqrt{2a_1(\delta a_1 - \delta a_4)}\right). \tag{7}$$

Since a_1, δa_1, and δa_4 are all negative, the perturbation δa_1 affects the damping, and δa_4 the frequency of the oscillation. Simultaneous measurement, however, may decrease both perturbations, depending on the value of $|\delta a_1 - \delta a_4|$. When $\delta a_1 - \delta a_4 = 0$, both perturbations disappear.

From the above analysis on a simple RLC circuit, it is possible to conclude that in some situations a simultaneous measurement approach can be more advantageous than a sequential one in terms of suppressing the perturbations on a system's dynamical behavior.

As demonstrated by the foregoing analysis, we are interested here in the *short-term* dynamical behavior of the RLC circuit. Such behavior is important in the context of real-time control applications, where the parameters of a system need to be monitored on a permanent basis and measured at regular intervals [7]. By contrast, it is clear that the *long-term* behavior of the RLC circuit (a *linear* dynamical system) is very simple: The circuit settles into a stable equilibrium state. Asymptotically, I tends to 0 and U_C tends to E.

3 Conclusion

A hypothetical physical system is described in [3] with the property that certain operations on its parameters can only succeed if performed in parallel. Instances of these operations include measuring or setting a number of physical attributes of the system, such as temperature, pressure, voltage, and so on. Success or failure of these operations is determined by the laws of nature governing the behavior of the system. Examples of these laws are *Heisenberg's uncertainty principle* in quantum mechanics, which puts a limit on our ability to measure with a high degree of accuracy pairs of 'complementary' variables, *Le Châtelier's principle* of chemical systems in equilibrium, and the *homeostatic principle* in biology which is concerned with the behavior displayed by an organism under stress.

In this paper we presented an example of such a system. A simple RLC circuit is described whose dynamical behavior is significantly affected by sequential measurements of its parameters. A parallel measurement approach, on the other hand, greatly mitigates these perturbations and often eliminates them altogether.

As mentioned at the end of Section 2, the RLC circuit under consideration in this paper is a *linear* dynamical system [5]. As such, the perturbations it experiences from measurement of its parameters are, in general, relatively small. It remains to see whether dynamical systems that are *nonlinear* suffer more dramatically from sequential measurements of their attributes. Furthermore, the result of this paper may be generalized in the sense of allowing more than two measurements to be performed in parallel.

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