Technical Report No. 2004-490
Parallel Computation and Avoidance of Chaos*

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December 6, 2004

Abstract

Many well known mathematical models display a transition from simple motion to chaos as parameters of the system change. Here we will look at a particular example, the forced damped oscillation model, and show that a parallel approach to changing parameters can help to control the behaviour of the system when it is in a near chaotic state.

1 Introduction

Parallel computation is an approach to information processing where many processors cooperate to solve a computational problem by working on it simultaneously. Traditionally, parallel computing was thought to be only a tool useful for speeding up computations which would take an unacceptable amount of time to execute on only one processor. However, recent results have shown that parallelism can not only increase the speed of a computation, but also improve the quality of the result. In addition to this, computational environments have been found where certain tasks will succeed if and only if they are performed in parallel. In particular, this paper builds on the results presented in [2], [3], and [4].

The three aforementioned papers exhibit a paradigm in which the analysis of a physical system is enhanced by parallel methods. A class of systems is described in [2], in which a parallel approach is necessary to obtain valid results, due to the way the parameters of a system are related. In [3] and [4], physical systems in equilibrium are described with the propety that measuring one of their variables modifies the values of any number of other variables unpredictably. As a result, the values held by all variables, save one, when the system is in a state of equilibrium are effectively lost, and any subsequent mesurement of these values is futile. It is shown in [3] and [4] that under these conditions a parallel approach succeeds in measuring all the values precisely, while a sequential approach fails.

The purpose of this paper is to address the following question presented in [4]: Is parallelism useful in systems whose equilibrium is disturbed when the values of their variables are deliberately modified (rather than measured) by an external agent? We answer this question in the affirmative by exhibiting a physical system which is forced into chaotic behaviour when its parameters are changed sequentially, but retains its stable behaviour when the changes are made in parallel. As in [3] and [4], we use a dynamical system to demonstrate our point.

A dynamical system, in general, is defined as a deterministic process in which a variable's value changes over time according to a rule that is defined in terms of the variable's current value [9]. Common applications of dynamical systems include population dynamics, electric circuits, and chemical reactions, to name a few. This paper will focus on a dynamical system known as the damped forced oscillation model. We first observe how the system behaves, and then show that in order to avoid pushing it into chaotic behaviour when making a change to its parameters, it is sometimes necessary to perform changes in parallel.

*This research was supported by the Natural Sciences and Engineering Research Council of Canada

Chaos is an important concept in the study of dynamical systems and it will figure prominently in our analysis. Traditionally, the term chaos evokes a notion of randomness, or continual change without any rhyme or reason. This is actually a common misconception, as chaotic behaviour is completely deterministic. There is, in fact, no element of randomness inherent in chaos. Chaotic behaviour is best described as sensitive dependence on initial conditions [8]. Chaos is neither random nor completely unpredictable. Nevertheless it presents a problem, as predicting real-world chaotic behaviour accurately requires more precise data than can possibly be obtained. This is why chaotic behaviour is so undesirable. Although the model of a real world system may be accurate, even an extremely short period of chaos is enough to cause the model to make incorrect predictions [1].

The remainder of this paper is organized as follows. Previous work on which this paper builds is summarized in section 2. Section 3 then introduces the dynamical system of interest. The behaviour of this system is studied in section 4 through a phase plane analysis. Section 5 shows the effects of the parallel and sequential approaches to changing its parameters. Section 6 examines the results of section 5 to introduce a more general corollary. Conclusions and an open problem are given in section 7.

2 Related Work

An important characteristic of traditional analyses of parallel computation is that the conditions governing the computational environment are, in a fashion, fully determined by the human in charge and the model of computation used. For example, in a real-time computation, if it is deemed that the arrival rate of the data is too high, it is possible for the people responsible for the computation to slow down the arrival rate, or to extend the deadline by which a solution is to be delivered, or to use a faster computer, and so on.

A radical departure from this paradigm was taken recently. In [2], the focus is on computational environments in which a computation can succeed if and only if it is performed in parallel. In these environments, it is the laws of nature that prevail, rather than human-imposed computational circumstances or conditions on the computation. Specifically, let $\mathcal{S}$ be a physical system, such as one studied by biologists (e.g., an ecosystem), or one maintained by engineers (e.g., a nuclear reactor). The system has n variables each of which is to be measured or set to a certain value. One property of $\mathcal{S}$ is that measuring or setting one of its variables modifies the values of any number of other variables of the system unpredictably. It is shown in [2] that under these conditions a parallel approach succeeds in carrying out the required operations on the variables of $\mathcal{S}$, while a sequential approach fails. Such computations are said to be inherently parallel *in the strong sense*, meaning that they are efficiently executed in parallel, but impossible to carry out sequentially. Furthermore, it is principles governing such fields as physics, chemistry, and biology, that are responsible for causing the inevitable failure of any sequential approach to solving the problem at hand, while at the same time allowing a parallel approach to succeed. A typical example of such principles is the uncertainty involved in measuring several related parameters of a physical system. Another principle expresses the way in which the components of a system in equilibrium react when subjected to outside stress.

Concrete examples of the system $\mathcal{S}$ were described in [3] and [4]. In [3], it is shown that a resistance-inductance-capacitance (RLC) circuit (a linear dynamical system), certain parameters of which are measured sequentially, that is, one after the other, undergoes significant perturbations that affect its dynamical behavior. By contrast, these perturbations could be eliminated when the measurements are performed in parallel, that is, when the parameters are measured simultaneously. In [4], the effect of measurements on the behavior of a nonlinear dynamical system, namely, the Belousov-Zhabotinskii chemical reaction (BZ-reaction), is analyzed. It is shown that measurement disturbs the equilibrium of the system and causes it to enter into an undesired state. If, however, several measurements are performed in parallel, the effect of perturbations seems to cancel out and the system remains in a stable state.

The examples in [3] and [4] illustrate what happens when the parameters of a dynamical system in equilibrium are measured, sequentially, compared with what happens when they are measured in parallel. In what follows we study the consequences of *setting* the parameters of a dynamical system sequentially and in parallel.

3 The Forced Damped Oscillation Model

The simplest way to understand the forced damped oscillation model is as a means to predict the motion of a pendulum under the influence of friction, gravity, and a sinusiodal torque force which pushes the pendulum either clockwise or counterclockwise. The way these forces act on the pendulum is illustrated in Fig. 1.

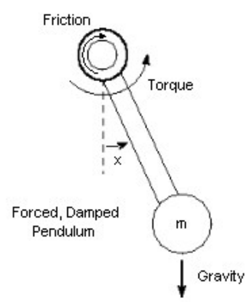


Figure 1: Diagram of the forced damped pendulum

Considering all of these forces, a second order differential equation can be obtained to describe the motion of the pendulum, where x is the position of the pendulum after t units of time [8]:

$$\frac{d^2x}{dt^2} + \mu\frac{dx}{dt} + \frac{g}{L}\sin x = A\cos(pt), \tag{1}$$

where

x - Position of the pendulum in radians

μ - Coefficient of friction

g - Gravitational constant

L - Length of the pendulum arm

A - Amplitude of the torque force

p - Period of the torque force

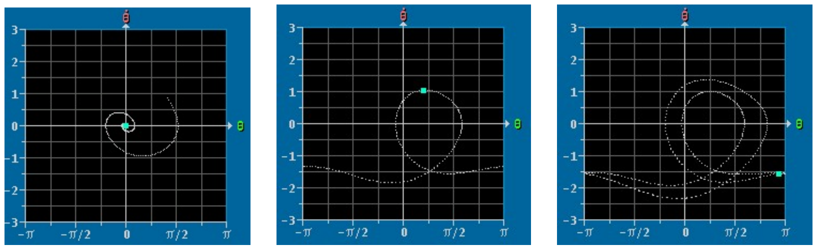
t - Time.

In order to learn more about the system, it is beneficial to go through a phase plane analysis, observing the way in which the system evolves as its parameters change. An important phenomenon to observe during the phase plane analysis is known as *bifurcation.* A bifurcation point is a point at which the period of a dynamical system's solution doubles [6] (or in this particular case, a point at which the period of the pendulum's oscillation doubles).

4 Phase Plane Analysis

By looking at the phase plane for different parameter values, we can determine the behaviour of the system, and analyze it. The parameter A will be slowly increased to show a gradual change of the system, ending in a chaotic state. Velocity is plotted on the vertical axis, and position is plotted on the horizontal axis. In all cases, let $g = 9.8$, $L = 9.8$, and $\mu = 0.5$ for simplicity.

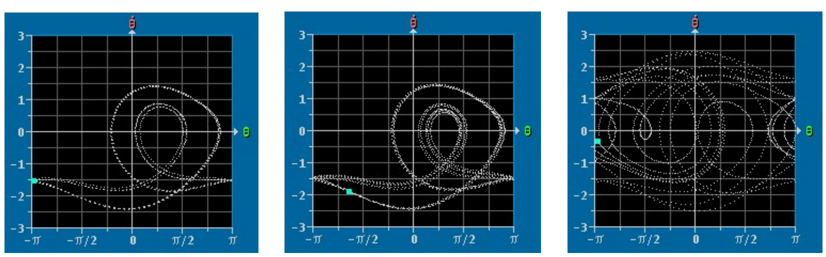
Initially, the parameters will be set so that $A = 0$, $p = 0.67$. In this case the torque force is 0, so the system is simply losing energy due to friction, and hence the point of minimum energy where position and velocity are both 0 is a stable point attractor, see Fig. 2(a). Adding a significant torque force, A becomes 1.35 while the other parameters stay the same. Now the system is gaining energy from the torque force as well as losing it due to friction. For these particular parameters the pendulum settles into a periodic single loop orbit in the phase plane. Let the period of this orbit be T. Note that this period-T orbit crosses the horizontal axis (the pendulum changes direction) at only two points, see Fig. 2(b). Increasing the torque force to 1.45 but again preserving the values of the other parameters, we can see the system's behaviour has changed. The system has undergone a bifurcation and the period of the oscillations has doubled. Now the pendulum is exhibiting a period-$2T$ orbit (note the two loops), and changes direction at four different points, see Fig. 2(c).



(a) $A = 0.00$ (b) $A = 1.35$ (c) $A = 1.45$

Figure 2: Phase plane diagrams for varying values of the parameter A

Continuing to increase the parameter A to 1.47, the system has now undergone another bifurcation, and the period of the oscillations has doubled again. The pendulum is now orbiting with period-$4T$, and changes direction at eight different points, see Fig. 3(a). The successive period doubling will eventually lead to the onset of chaos. Increasing A yet again to a value of 1.48, we can start to see chaotic behaviour emerging, see Fig. 3(b). Note that it is not clear how long the period of oscillation has become. In fact, if one were to zoom in on a trajectory in the phase plane, a fractal structure would begin to emerge, where every trajectory is in fact many closely spaced trajectories, and each of those is comprised of many even more closely spaced trajectories. Increasing A to a value of 1.50 causes the system to behave very chaotically, see Fig. 3(c). If the simulation were to continue running indefinitely, almost all of the phase plane would be coloured white by trajectories. Note that we only need to increase the amplitude of the torque force slightly, from 1.47 to 1.50, to go from stable period-$4T$ oscillation to complete chaos.



(a) $A = 1.47$ (b) $A = 1.48$ (c) $A = 1.50$

Figure 3: Phase plane diagrams for varying values of the parameter A

5 Changing The Parameters Of The System

The best way to see where the successive period doubling and transition to chaos occurs is a bifurcation diagram. In the diagram, the number of points along any vertically drawn line represents the period of an orbit in the phase plane over a theoretically infinite period of time as a function of A, the amplitude of the forcing term. For example, if we draw a line straight up from $A = 1.45$ there are two points on the graph that the line will pass through. Hence, the solution to this system will have a period-$2T$ orbit for that particular value of A.

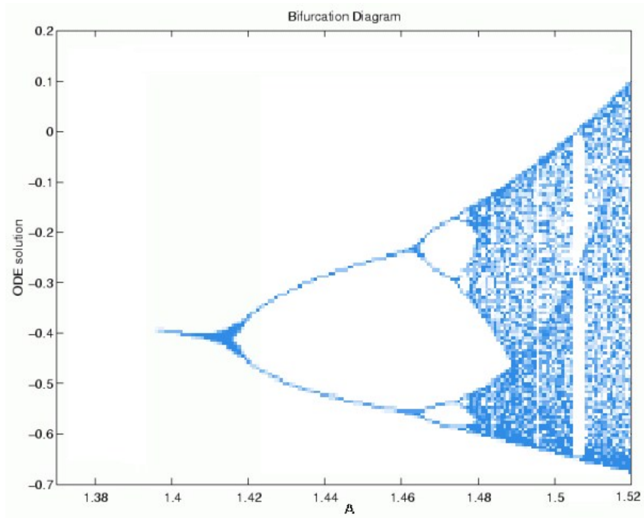


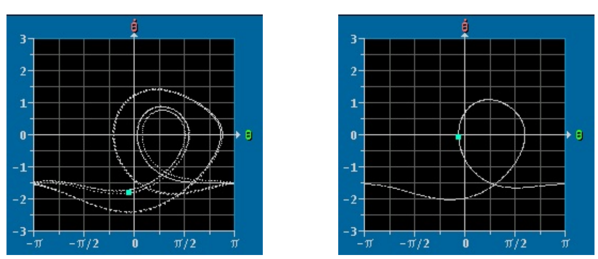
Figure 4: Bifurcation diagram for the forced damped pendulum

Figure 4 shows the sensitive region for the amplitude term, values between 1.4 (period-T oscillation) and 1.5 (complete chaos). This is the area we are interested in, the so called 'edge of chaos'. The main concern of the following section is with avoiding a transition through a period of chaotic behaviour. In the first scenario, two parameters will be changed in parallel, and in the second they will be changed in sequence. The first scenario displays simple behaviour by avoiding a transition through chaos, while the second displays less predictable behaviour because the system must undergo a short period of chaos.

5.1 Scenario 1 - Parallel Changes

In this scenario, the pendulum will start in a stable period-$4T$ orbit, with parameter values at $A = 1.47$, $p = 0.67$, $\mu = 0.5$. After settling into a stable behaviour, suppose that for some reason the system parameters need to be changed so that $A = 1.50$, $p = 0.72$, $\mu = 0.5$. Additionally, suppose that for some reason inherent to the system being studied, the amplitude, A, of the torque force cannot be changed after the period, p. The changes will be made in parallel, in order to preserve as much as possible the predictable behaviour of the system.

The system starts in a stable period-$4T$ orbit, with the parameter values as indicated previously, as seen in Fig. 5(a). Now the parameters A and p are changed at exactly the same time from 1.47 to 1.50, and 0.67 to 0.72 respectively. The system goes directly from its previous state to the new state. Here we see simple period-T oscillation, the system reaches this new state quite quickly, see Fig. 5(b).



(a) Initially (b) After A and p are changed

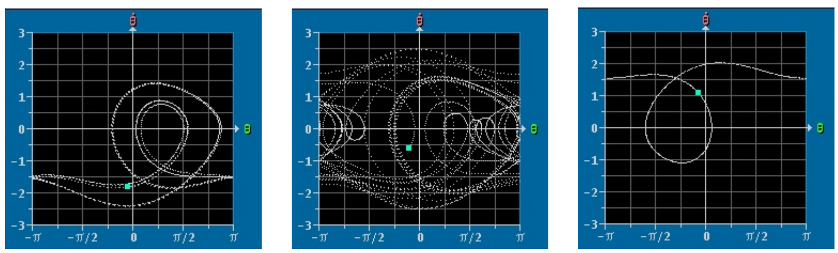
Figure 5: Phase plane diagrams before and after parallel changes

The parallel changes cause the system to move quickly from its former period-$4T$ behaviour to a new state of period-T oscillations. There is no point at which the pendulum displayed chaotic behaviour at any time during the change. Hence, the result of changing the two parameters is simple and predictable.

5.2 Scenario 2 - Sequential Changes

As in the first scenario, the pendulum starts at a stable orbit with parameter values $A = 1.47$, $p = 0.67$, $\mu = 0.5$, and the state of system needs to be changed, for whatever reason, so that the parameters satisfy $A = 1.50$, $p = 0.72$, $\mu = 0.5$. As before, suppose the amplitude, A, cannot be changed after the period, p. Here the parameters will be changed one at a time, starting with A, the amplitude of the forcing term, and then p, the period of the forcing term.

The system starts in a stable period-$4T$ orbit, with the parameter values as indicated above, see Fig. 6(a). After letting the system run with the previous set of parameters and settle into a stable state, the amplitude of the forcing term, A, is incremented to 1.50, leading to chaotic behaviour, exhibited in Fig. 6(b). After a brief period of chaotic behaviour, the period of the forcing term, p, is incremented to 0.72. The system has the same parameter values as it did at the end of scenario 1, but now the oscillation has changed direction, as seen in Fig. 6(c).



(a) Initially (b) After A is changed (c) After p is changed

Figure 6: Phase plane diagrams before, during, and after sequential changes

Compare the trajectories in the phase plane at the end of scenario 2 to those at the end of scenario 1, Fig. 6(c) and Fig. 5(b) respectively. The system is in stable period-T oscillation in both cases, but in scenario 2 it has settled into an attractor in a different part of the phase plane. This is due to the short interval of time in which the system is chaotic. Depending on where the pendulum is when the amplitude of the forcing term is changed, the system may diverge to unstable behaviour extremely quickly. It may not be possible to tell beforehand which of the two attractors the system will stabilize at, once the short period of chaos is over [7]. This is where the main advantage of making parallel parameter changes is seen. It is possible to predict accurately before changes are made, exactly what will happen to the system, since there is no period in which chaos occurs. Note also that, although not shown here, the pendulum reaches its new stable orbit extremely quickly when parallel changes are made. Sequential changes, by contrast, can cause the transition period from chaos back to a stable orbit to be significantly longer.

6 Why Does This Occur?

Comparing the parallel to the sequential approach, it is clear that in each case the results are different. But how is it possible to tell, given a dynamical system, when this will be the case? Recall, in the system of interest here, the parameters that were chosen to be changed, that is, the amplitude of the periodic force, and the period of that force. Creating a simple graphical plot of the stability (that is, the periodicity of solutions) of the system as a function of both of these parameters is a good way to illustrate why the parallel approach succeeds. In Fig. 7, the horizontal axis represents amplitude, while the vertical axis represents period. Gray areas represent chaotic behaviour, and white areas represent periodic behaviour.

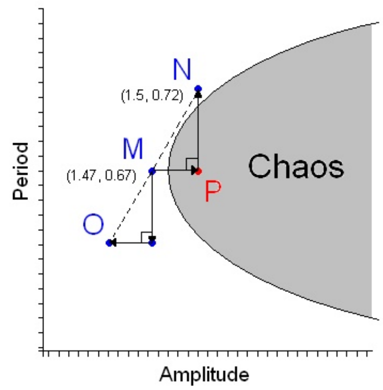


Figure 7: Periodicity of solutions as a function of parameters A and p

Figure 7 shows changes in the parameters of the system. The scenarios presented earlier involved moving from point M to point N. By changing first the amplitude, and then the period of the forcing term, some intermittent amount of time is spent in a region in which the system is chaotic. This is why a parallel approach is advantageous. Using parallelism, the system parameters go from point M to point N directly, without having to go through point P. In general, dynamical systems that are at the edge of chaos will benefit most from the parallel approach. If the system parameters had been changed from what they initially were at M to a point which moves away from the chaotic region, such as point O, then the system's future behaviour is very simple and predictable even when multiple parameters are changed sequentially. The essence of the parallel approach lies in the fact that the system is very close to a region in which its behaviour is chaotic, or even completely unpredictable.

7 Concluding Remarks

It has been shown that parallelism is an approach that may be of value in the field of dynamical systems control, by presenting one particular simple example, the forced damped pendulum. In section 5, the difference between parallel and sequential approaches to altering parameters was illustrated. The sequential approach resulted in pushing the system through a short period of chaos, thus adversely affecting the validity of any results. However, note that in Fig. 7, if the period was changed first, and then the amplitude, the region of chaotic behaviour could be avoided, as is shown in Fig. 8. This way, sequential changes could yield a result as predictable as the parallel approach. As was assumed earlier, there could be some real-world restriction on parameters of the system, such that we cannot alter the amplitude after the period has been changed, but more than likely this is not the case. A topic for future research is therefore to find a dynamical system where parallel changes are absolutely necessary in order to get predictable results. Such a system may have a stability plot much like that shown in Fig. 9.

Acknowledgements

The first author would like to thank Dr. Daniel C. Offin for proposing the differential equation model of forced damped oscillation as a dynamical system of interest.

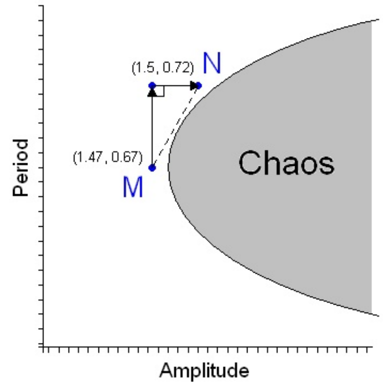


Figure 8: Avoiding chaotic behaviour by reversing the order of parameter changes

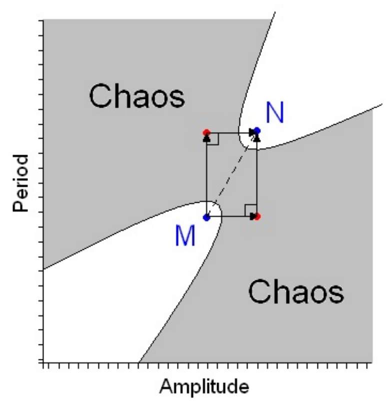


Figure 9: A system where parallel parameter changes are absolutely necessary

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