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Why there is no such discipline as hypercomputation

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The editors have kindly invited me to write an introduction to this special issue devoted to "hypercomputation" despite their perfect awareness of my belief that there is no such subject. In [5], I rather thoroughly debunked the claims of two of the leading proponents of this supposed branch of learning. In this brief introduction I will content myself with some general remarks.

The analysis of algorithmic process that emerged from the work of Gödel, Church, Turing, and Post has been of great importance not only for theoretical investigations but also for practice, by providing an expansive framework for computer science. The discussion of computation-like processes that transcend the limits imposed by the Church–Turing thesis can likewise be framed either in terms of theory or of practice. However, although the term "hypercomputation" is quite recent, the theoretical side has been flourishing for over half a century, and for the most part without the benefit of those contributing to this issue. As for suggestions in the realm of practice, they boil down to exhortation ("The search is on") and to the trivial insight: if a physical process exists that generates a definite non-Turing computable function or real number, then such a process could be used to compute the non-Turing computable [2]. And even this triviality is questionable, as will be explained, because all that we experience is finite.

1. Hypercomputation as practice

The great day has arrived! The world's first working hypercomputer is being unveiled. Here comes the output – guaranteed by the engineers to be a non-Turing-computable infinite sequence of natural numbers:

$$23, 5, 1267, 111, 59, 87654, 21, 1729, 88888881, \text{etc.}$$

But wait! No matter how long this goes on, we will see only a finite number of these outputs. Moreover any such finite sequence of natural numbers is the initial part of both computable and non-computable infinite sequences (in fact, infinitely many of each kind). Thus, no finite amount of data will suffice to distinguish the computable from the non-computable, and since we, as finite beings with finite lifetimes will only have access to a finite amount of data, no possible experiment could certify that a device is truly going beyond the Turing computable.

So, on what basis can someone claim that some device is indeed a "hypercomputer"? It can only be on the basis of physical theory. Such a theory would have to be certified as being absolutely correct, unlike any existing theory, which physicists recognize to be only an approximation to reality. Furthermore, the theory would have to predict the value of some dimensionless uncomputable real number to infinite precision. Finally, the specifications of the machine would have to guarantee that it exactly follows the demands of this supposed theory. Needless to say, nothing like this is even remotely on the horizon.

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Given all of this, what is one to make of the claims of Tien Kieu [8,9] to the effect that the Quantum Adiabatic Theorem can be used to devise a probabilistic algorithm for solving Hilbert's tenth problem, known to be unsolvable. Hilbert's tenth problem may be stated as follows:

Find an algorithm that will determine of any given equation

$$p(x_1, x_2, \ldots, x_n) = 0, \qquad (1)$$

where p is a polynomial with integer coefficients, whether the equation has a solution in positive integers.

This is not the place for a detailed discussion of Kieu's rather intricate arguments involving matters concerning which I have very limited expertise. However, it may not be amiss to mention that in conversation with some of the leading experts in quantum computation, including experts in the use of the Quantum Adiabatic Theorem, I was assured that there is no way that these methods enable one to accomplish the miracle of surveying infinitely many tuples of natural numbers in a finite time. But even if Kieu is correct, and these experts are wrong, and the Adiabatic Theorem does imply that such an infinite search is possible, I would adopt the classic stance of Hume regarding miracles. Bearing in mind that existing physical theory can only hope to approximate reality, I would regard such an implication as calling into question the Quantum Adiabatic Theorem itself.

Andrew Hodges posted a critique of Kieu's ideas on the FOM (foundations of mathematics) email list [7]. Although Kieu is a subscriber to the FOM list, and has posted on it, he does not seem to have responded to Hodges comments which deal precisely with the "magic" of such an infinite search. However, Kieu did remark as follows on FOM [10][1]:

> I have nothing more to add to what I have already said about energy measurement here (except to repeat that the final spectrum is integer-valued so the gap is 'at least' one unit in energy, which is then sufficient for specifying DE without knowing the spectrum). However, I would like to point out that beside the energy measurement we can also do with the measurement of occupation number (represented by the operator $a^\dagger$ a), since by construction the end-point Hamiltonian (H_P in my notation) commutes with the number operators so these two observables (energy and occupation number) are compatible. Such occupation number measurement would give us the positive integers n_1, n_2, etc., which we can then substitute into the Diophantine polynomial in the places of corresponding variables to see if the polynomial vanishes (in which case, the equation has some positive integer solution) or not (no integer solution). This point has been made in, for example, my paper in Contemporary Physics.

Now, obviously there are Diophantine equations that do have solutions but where the numbers constituting their least solution are all greater than

$$10^{10^{10^{10}}}$$

It is all very well for Kieu to speak of substituting such numbers "in the places of corresponding variables to see if the polynomial vanishes". But these are numbers larger than the number of electrons in the observable universe. In what sense would it be physically possible to read off such numbers from apparatus? The simple arithmetic required to do the indicated calculation of the value of the polynomial would likely take longer than the expected life of the solar system. So at best, Kieu's method could only hope to test for the existence of solutions less than some constant N, a limit of feasibility. But obviously quantum mechanics is not needed for this. The *finite* search for solutions bounded by such a constant is of course classically decidable.

The two pillars of contemporary physics are quantum mechanics and relativity theory. So it was inevitable that relativity theory would be brought to bear on solving the unsolvable. In [6], Etesi and Nemeti argue that conditions in the vicinity of certain kinds of black holes in the context of the equations of general relativity indeed permit an infinite time span to occur that will appear as finite to a suitable observer. Assuming that such an observer can feed problems to a device subject to this compression of an infinite time span, such a device could indeed solve the unsolvable without recourse to Kieu's miracle of an infinite computation in a finite time period. Of course, even assuming that all this really does correspond to the actual universe in which

[1] [10]. Correction: In this item, Abdel Perez-Lorenzana's name is incorrectly given as "Abdul Lorenz".

we live, there is still the question of whether an actual device to take advantage of this phenomenon is possible. But the theoretical question is certainly of interest.

2. Hypercomputation as theory

While one may justly be skeptical regarding practice, non-computability (in the Church–Turing sense of course) is an inviting subject for theoretical study. Although this may be news to some of the promoters of "hypercomputation", this has been an extremely active field of research beginning with the work of Post and Kleene in the 1940s. In particular Post's Theorem makes it possible to relate just how non-computable a particular problem is to its logical complexity. For example if membership in a set S of natural numbers is computable relative to the set of halting Turing machines as an oracle then one may write:

$$n \in S \Longleftrightarrow \forall x \exists y_1, y_2, \ldots, y_k [p(n, x, y_1, y_2, \ldots, y_k) = 0],$$

where p is a polynomial with integer coefficients. In addition a scale of degrees of unsolvability has been developed and studied providing a means of measuring just how unsolvable a particular problem is. This has been a vast effort involving perhaps fifty researchers over a period of over half a century.

Also there is a rich literature concerning computation over such objects as functions, real numbers, and transfinite ordinals.[2]

3. Computers and computability

Although real-world computation is finite and computability theory deals with the infinite, the latter has had an important influence on the former. It was Turing's work that has played the key role. His universal machine and the Church-Turing explication of the notion of algorithmic process together suggested the possibility of an all-purpose computer that could execute any algorithm subject only to constraints of space and time. Furthermore, as I have remarked elsewhere:

> Before Turing the general supposition was that in dealing with such machines the three categories, machine, program, and data, were entirely separate entities. The machine was a physical object; today we would call it hardware. The program was the plan for doing a computation, perhaps embodied in punched cards or connections of cables in a plugboard. Finally, the data was the numerical input. Turing's universal machine showed that the distinctness of these three categories is an illusion. A Turing machine is initially envisioned as a machine with mechanical parts, *hardware*. But its code on the tape of the universal machine functions as a *program*, detailing the instructions to the universal machine needed for the appropriate computation to be carried out. Finally, the universal machine in its step-by-step actions sees the digits of a machine code as just more *data* to be worked on. This fluidity among these three concepts is fundamental to contemporary computer practice. A *program* written in a modern programming language is *data* to the interpreter or compiler that manipulates it so that its instructions can actually be executed.[3]

One sometimes sees comments to the effect that it is false to claim that real-world computers are bound by Church's Thesis, or to claim that they only compute Turing-computable functions. In a strict sense, any such claim would be senseless – real-world computers are bound by a much stricter constraint: everything they do is finite. What is true is that computers are programmed to execute algorithms. Although programmers are forced to be aware of the finite bounds that constrain their work, it has been found useful, even for practical purposes, to study algorithms defined as applying to inputs of arbitrary size. The asymptotic behavior of algorithms that is taken as a measure of their complexity is an example of this. For further discussion see [5].

[2] For surveys of some of these matters, see [1] Chapters C.4–C.8.
[3] [3,4, pp. 164–165].

4. The hypercomputation community

Whatever one may think of the scientific value of the efforts of the founders of hypercomputationalism, there is no question that as promotion it has been a brilliant success. Ignoring what had been accomplished in the study of non-computability, and turning the negative into a positive by replacing the ''non'' by ''hyper'' was surely an inspired move. Examining the table of contents of this special issue suggests that although a number of the articles have little or no connection to the project of computing the non-computable, their authors have thought it appropriate and worthwhile to place their work under this questionable banner.

But let me end this introduction on a positive note. Toby Ord, a philosopher, and Tien Kieu, a physicist, have been brought together precisely by their common interest in hypercomputation. And they have recently obtained a neat result (later generalized by Yuri Matiysevich) concerning Gregory Chaitin's famous number Ω, but with no evident connection to hypercomputation [11]. Namely they have found a particular exponential Diophantine equation $T(k, z_1, \ldots, z_m) = 0$ such that for every k this equation has finitely many solutions in $z_1, \ldots, z_m$ and the number of solutions is odd if and only if the kth digit of Ω is equal to 1.

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