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Hypercomputation by definition

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Abstract

Hypercomputation refers to computation surpassing the Turing model, not just exceeding the von Neumann architecture. Algebraic constructions yield a finitely based pseudorecursive equational theory (Internat. J. Algebra Comput. 6 (1996) 457–510). It is not recursive, although for each given number n, its equations in n variables form a recursive set. Hypercomputation is therefore required for an algorithmic answer to the membership problem of such a theory. Yet Alfred Tarski declared these theories to be decidable. The dilemma of a decidable but not recursive set presents an impasse to standard computability theory. One way to break the impasse is to predicate that the theory is computable—in other words, hypercomputation by definition.

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Preface

A. I. Mal'tsev's work inspired the hunt for pseudorecursive varieties, and Alfred Tarski's encouragement led to publishing the findings. Although the heir of a two-figure fortune, I am still a beginner at transcending duality.

1. Introduction

We consider here computability problems on the boundary between logic and algebra. For simplicity we fix the algebraic type as one binary operation + and two individual

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constants a, b. Using these symbols and an infinite supply of variables v_i ($i \in \mathrm{N}$, the set of natural numbers), we can specify logical *terms* as algebraic expressions, such as $v_1 + v_2$ or $b + (b + v_{17})$. A *formal equation* (or *identity*) is a string composed of two terms separated by the formal equality sign $\approx$, such as $a + v_3 \approx v_3 + a$. An *equational theory* of this type is a set T of formal equations including the equation $v_1 \approx v_1$ that is closed under these two operations: (a) replacing a subterm t_1 appearing in an equation in T by a term t_2 when $t_1 \approx t_2$ or $t_2 \approx t_1$ is in the set T, and (b) substituting a chosen but arbitrary term for every occurrence of a variable in an equation in T. In high-school algebra, simplifying and factoring are processes that use these two rules and the identities called the laws of algebra, such as associativity, commutativity, and distributivity. A subset B of an equational theory T is an *equational base* for T iff T is the smallest equational theory that includes B. We write $T = \mathrm{Th}(B)$. Thus T is *recursively based* just in case T is the closure under (a) and (b) of a finite set, or an infinite recursive [1] set, of equations. The class of algebraic models for an equational theory is called its *variety*; thus we also speak of finitely based varieties.

All examples that we consider are equational theories for varieties of *semigroups*; that is, they contain the equation

$$(v_1 + v_2) + v_3 \approx v_1 + (v_2 + v_3) \qquad (*)$$

that guarantees associativity for the $+$ operation. They may also provide a *zero* element by containing the equations $0 + v_1 \approx 0 \approx v_1 + 0$; here, 0 may be a new individual constant or a particular term. To continue the analogy with high-school algebra, we are concerned with validity of algebraic laws such as commutativity rather than finding solutions of algebraic equations such as $x^2 = 2$. A useful equational theory of semigroups in computer science is the theory whose only equations are the instances of $v_1 \approx v_1$ and associativity $(*)$. This is the theory of nonempty strings over an alphabet of two letters, also called the theory of the free semigroup on two generators. It has a finite base, consisting of $(*)$ alone.

Let T_n be the subset of an equational theory T consisting of the equations in T in which no more than n distinct variables appear. Then T is *quasirecursive* iff for every number n, T_n is recursive. T is *pseudorecursive* iff T is quasirecursive but not recursive. Note that if T is recursively based (in particular, if it is finitely based), then T will certainly be recursively enumerable, i.e., listable by a Turing machine (TM).

Various pseudorecursive equational theories were constructed in [45,47] and extended in [49,51]. According to [47], if we start with a fixed but arbitrary nonrecursive, recursively enumerable set $X \subset N$, we can define a finite equational base $\Psi 1_X$ from a highly engineered TM that accepts X. The resulting theory $\mathrm{Th}(\Psi 1_X)$ is an equational theory of semigroups with zero and finitely many individual constants (the number of constants can be reduced to two or one [49], or eliminated [45]). The main result is quoted here.

[1] More traditionally, the set of Gödel numbers of these equations is recursive; that is, its membership can be decided by a Turing machine.

Theorem 1.1 (Wells [47], Theorem 10.4). *For every nonrecursive r.e. set* $X \subset N$, $\mathrm{Th}(\Psi 1_X)$ *is a finitely based pseudorecursive equational theory*; *indeed*, $\mathrm{Th}(\Psi 1_X) \equiv_{\mathrm{m}} X$.

I shared the problem of finding a finitely based pseudorecursive variety with Alfred Tarski in 1970. He remained excited about it for a decade, even when I was not. As noted in [45,47,48], he took the result as an argument for the necessity of formal language, not only in mathematics, but even in logic itself. Logic discusses formal languages, but Tarski felt that the use of exact discourse was, paradoxically, often neglected during that discussion. In 1982, he asked me why I thought he liked the discovery/construction of finitely based pseudorecursive theories that I had finally written into a dissertation. It turned out he did not have in mind the oft-repeated indispensability of formal language. Instead, he astonished me by asserting: "Because they are decidable." In other words, a nonrecursive set was decidable. Did that mean he thought it was computable? As he described, there would at least be a finitistic, mechanistic method for making the decision.

Tarski's statement is quoted in [45, p. xii–xiv; 47, p. 460–461]. Wells [45–47] introduce a discussion of his controversial claim that these nonrecursive theories are decidable and so appear to contradict the Church–Turing Thesis (CTT); Wells [48] elaborates. [2] The current article, enlarging on Wells [50], discusses the background for simply declaring finitely based or, more generally, r.e. pseudorecursive varieties to be computable (see [48, Section 9]). A mathematical framework for this discussion will appear in [52].

The following result strengthens Theorem 1.1 inter alia and is needed below. Let P be the class of problems solvable by deterministic TMs halting in time bounded by polynomial functions of the length of the input.

Theorem 1.2 (Wells [51], Theorem 3.1). *Each pseudorecursive theory* $\mathrm{Th}(\Psi)$ *discussed in* [45, *Section* 10] *and* [49] *has* $\mathrm{Th}_n(\Psi) \in \mathrm{P}$ *for every* $n \in N$.

2. Mathematical impasse

Let us consider some examples of how mathematics expands.

First, mathematics solves open problems. For some problems, this can take a long time—witness the recently solved Fermat's Last Conjecture and the unsolved Riemann hypothesis. Sometimes the first solution proves inadequate. The second may involve a detailed reformulation of previously announced results that presented flaws or initially unappreciated imprecision. Instances of this include the publication of the details of Tarski's decision procedure for real-closed fields in [42] after its elliptical

[2] Tarski's claim immediately concerned the finitely based pseudorecursive theories (it also applied to all recursively based or merely r.e. pseudorecursive theories). He did not mean that the uncountably many pseudorecursive theories of Theorems 4.6–7 below and Remark 11.2 of [47] are all decidable. This point is contextually clear but not explicit in [48]. Curiously, his heuristic argument for decidability, given in [48, Section 5], applies equally well to the constructions from Theorems 4.6–7. This underlines its heuristic character.

announcement in [43, Chap. 6], and the revision of Mal'tsev's partially enunciated results in Chap. 3 of [31] as Chap. 11 there. [3]

There is also another phenomenon in mathematics that demands an answer; it may be termed an impasse because it asks more for a direction of escape than a solution. There are two chief approaches to enlarging knowledge in mathematics when confronted by an impasse.

If the cause of the impasse is an unsolvable problem or construction that carries beyond the domain in question, then *extension* is often employed as a resolution. In many cases, the extension solves the problem by assuming the problem can itself be abstracted as the answer and moving to the consequences of such a commitment. This happens when integers are introduced to solve $0 - n$, rational numbers to solve $1 \div n$ (or $1/n$), algebraic numbers to solve $\sqrt{n}$, and imaginary numbers to solve $\sqrt{-n}$. One approach offered in the current context is to assume that pseudorecursiveness is itself a new kind of computability; see Sections 4 and 5 below.

When the impasse is caused by a confrontation with accepted foundations and results, then mathematicians often seek to refine the context of discussion in order to provide a harmonizing resolution that offers insight into both sides of the conflict while disarming it. The lamination at the saddle of the Lorenz attractor [4] is a suggestive image of how a monolithic barrier has more ins and outs than can at first be conceived. With the picture of this fractal attractor in mind, let us refer to resolution of this impasse as *delamination*. Cases where turbid proposals have settled out as minable strata of mathematics include: Brouwer's "intuitive" intuitionism as clarified by BHK semantics and further formalization (see, e.g., Artemov [2]); the Skolem paradox as exemplified by inner models (see, e.g., [26]); set theoretic antinomies as resolved by types and new axiomatizations (for recent work, see [34]); the problem of individuals as explicated by beta models [30] and reinvigorated in Jensen's NFU (see [24]); the legitimation of infinitesimals in calculus as achieved by nonstandard models; [5] and semantics for the lambda calculus as provided by Scott's function spaces. [6]

[3] In a similar vein, the series starting with [47] is intended to bridge the lacunae of [45].

[4] The Lorenz attractor resembles a butterfly, but has nothing to do with the so-called butterfly effect associated with Ed Lorenz's famous comment on sensitivity to initial conditions. The attractor is formed from a complex, fractal arrangement of surface patches laminated in a manner resembling fillo pastry. Orbits of the system near the attractor fly around one or the other of the two "wings" in apparently random order, passing repeatedly through the saddle between them. Nearby orbits will quickly diverge into unrelated sequences of wing circlings. These traces imply an incredibly folded, ramified structure at the saddle. See, e.g., Peitgen et al. [36], but animated films and numerous applets on the Web give the best impression of orbits near the attractor. For nearly four decades, this shape was visualized but lacked proof of existence. That is, there was no mathematical demonstration that solutions of Lorenz's differential equations can be chaotic and that the attractor actually takes this fractal form known only through computer visualization. Finally, Tucker [44] proved it, using a computer; see also Stewart [41].

[5] Robinson [39] introduced nonstandard analysis, but it was some years before this ancient vision of calculus appeared in modern textbooks; see, e.g., [21].

[6] The original Scott semantics for the lambda calculus based on the absolute neighborhood retract of a function space was announced at the 1969 American Mathematical Society winter meeting in Los Angeles. Later developments in the semantics of the lambda calculus led to Scott–Strachey denotational semantics for computer programs (see [1,28]).

But there appears to be a third kind of impasse: when the obstruction is neither the novelty of the first impasse nor the superfluity of the second (that is, no answer or too many answers), but the matter exhibits profound vagueness and impenetrability, or internal self-contradiction, perhaps absurdity. A helpful metaphor may be that the first impasse is darkness, the second a kaleidoscope, and the third a duststorm. All can benefit from light—that is, elucidation. In the last type, patience can be rewarding. As an author refines his or her position and as the readers broaden their perspectives, the radical proposal can become acceptably familiar, grounded in realistic models, and producing insight and novel, useful results. [7] Of course, the proposal may be simply and resoundingly discredited; even then, there is often a history of fruitful revisitation. We can see that happening in studies of how close one can come to solving "unsolvable" problems: e.g., the minimal tools that suffice to trisect an angle, solve a quintic equation, prove Peano arithmetic consistent. [8] In time, the impasse may evolve into one of the two earlier types. Real number computability (unrealistic when uncountable), [9] symbolic integration theory (confusing), [10] the Liar paradox (self-contradicting), [11] ultraintuitionism (compelling, but profoundly vague) [12] can be argued to have made the transition. There is no suggestion that these programs have any

[7] A version of this stumbling block takes the form of the negation of unprovable but attractive propositions now shown to be independent. Thus, denying the parallel postulate, the axiom of choice, the axiom of regularity, and so on, leads to alternative models and mathematics. There is scope not only for conflicting views but also for provably incompatible visions of the true foundations of some area of mathematics. For example, Gödel held the Axiom of Choice to be true while intuitionists dismissed the theory of cardinals. He must have had some regard for the Axiom of Constructibility; many descriptive set theorists (some in California) support an incompatible form of the Axiom of Determinacy.

[8] Kalmár's proof of Gentzen's theorem reveals the precise requirement of ε_0-induction, the weakest possible tool beyond Peano arithmetic [37].

[9] The first efforts to treat real numbers in recursive function theory centered on specifying how to compute numerical operations when one had computations for their inputs. An advanced form of this approach is given by Mazur [33]. But Blum et al. [4,5] suggested a solution by extension: take infinite precision real-number computers as existing, then see what follows concerning their effectiveness and programmability.

[10] After implementations of promise in the early 1960s, notably Slagle's SAINT and Moses' SIN symbolic integration programs at MIT, the big theoretical breakthrough awaited Risch [38], which both inspired and confounded. Despite the ubiquitous success of Risch's analysis in symbolic algebra packages such as *Mathematica*, there remained a quest to extract efficient integration algorithms from practical heuristics and theoretical decision procedures (see [12,17] for the work of two decades). Risch proposed more than has been implemented, maybe even understood. Moses [35] gives an inside view of the early history of symbolic integration. Risch began his work on the integration of elementary functions by tackling a related problem assigned to him by Tarski. When Risch first presented his results at the UC Berkeley Logic Colloquium, Tarski objected that he had worked on the wrong problem. Users of *Mathematica*, etc., would disagree; consult Wolfram [53].

[11] Barwise and Etchemendy [3] give two cogent and parallel accounts of the Liar using situation logic and nonwellfounded sets, which violate Zermelo's axiom of regularity guaranteeing no set can be a member of itself.

[12] Esenin–Volpin [15,54] proposed a radically finitistic set theory with the unusual properties that $10^{10\wedge 10}$ is infinite and that it is possible to prove the consistency of ZFC. But until Geiser and others grounded it in Ehrenfeucht *-models, modal semantics, fuzzy logic, and other models, there did not seem to be a consistent way to envision it. See [14,18,25].

negative aspect or indeed anything in common except for falling under the current rubric. The first two have been handled by extension, the second two by delamination; all four cases still offer research opportunities.

Returning to the second form of impasse, we can identify typical tools of delamination as *relativization*, *specialization*, and *alteration*. All seek to restrict the overly general claims of the original problem or issue to more amenable contexts. Relativization results when there is access to additional information, usually about an interesting subproblem not yet solved (and usually unsolvable). A common form of relativization in the theory of computability provides a TM with an oracle in the form of an extra read-only tape. A typical oracle is the listing of a nonrecursive set in numerical order. [13] The source of the tape is unspecified and not in question. Observe that an appropriate oracular tape can make the decision procedures for the T_n uniform. Thus the question of a decision procedure for a finitely based or even just r.e. pseudorecursive theory (called a "Tarski procedure" in [48]) can be classically reduced to the real or physical existence of a suitable oracle. Some think that Turing may have speculated on the possibility of real oracles; see Copeland et al. [11]. This direction goes beyond mathematics and probably beyond computer science, so no more will be said here.

Specialization involves restricting the focus of the problem within its original domain. This often happens in preliminary results in mathematics: the theorem is first proven true for special cases. The question of the Tarski procedure can be recast for problems at a lower complexity than recursive functions, such as the class P; see [48, Section 9]; [51, Sections 2 and 4]; [52, Sections 2 and 3]; and Sections 3 and 4 below. Another type of specialization would be to restrict the algebraic properties of the underlying equational theories. This might result in a proof that pseudorecursiveness is impossible in that particular subdomain, eliminating there the search for Tarski procedures.

Let's use alteration to mean restating the problem for a context disjoint from the current setting. Frequently, first results do not apply directly to the original scope of the problem; for example, Julia Robinson (see [13]) showed the class of exponential diophantine problems to be unsolvable, but this provided energy for further study and foundation for Matiyasevich's resolution of Hilbert's tenth problem [32]. The Tarski procedure's arena is the class of r.e. sets and standard TMs. Moving beyond that setting, we can investigate elaborated or expanded domains and machine capabilities; see Section 4 below. These studies may in turn cast light on the original quandary.

In the earlier discussion [48] of the character a decision procedure for a pseudorecursive theory might assume, an attempt was made to analyze a hypothetical Tarski procedure in the light of Cleland's perspective on quotidian procedures (see also [9,10]). We can now go a step farther: the problem of formalizing quotidian procedures is an impasse of the third type probably evolving into the second type, while the problem of understanding the decidability of finitely based pseudorecursive theories should be clas-

[13] The classic delamination by relativization is the stratification of Turing degrees of unsolvability by means of the jump operator, recently shown to be an intrinsic property of the degree poset (see [27]).

sified as an impasse of the first type. [14] Further discussion of quotidianness is beyond the scope of this article.

3. Four directions

Tarski claimed that decidability should not be identified with recursiveness. For logic, the interest is whether this is necessarily vacuous or possibly a meaningful extension. From the point of view of computer science, however, an extension of the notion of computable function has little interest if there is no corresponding expansion of models for that computation. In other words, a pure existence proof of added computability carries little weight, much less an increase in computability by fiat—that is, hypercomputation by definition. To help overcome this, Wells [48, Section 9] identified four areas for investigation with the corresponding computing regimes (parenthetical references to terms in Section 2 have been added):

(1) the most general level of abstract computation, as traditionally understood; this is represented by TMs and directly related to the logician's concern with recursive functions (Extension);
(2) a lower level of complexity as determined by restrictions on time or space or other resources, including applications to the P/NP boundary (if any) or below (Specialization);
(3) variants based on modern machine capabilities and architectures as exemplified by parallel processing, and on
(4) for now unrealistic extensions of the theory of computation, in particular to real numbers (Alterations).

Concerning (1), the Church–Turing Thesis has met the test of time, a span a little short for mathematical history, but long in terms of the history of computing. According to widely accepted views, there appear to be no computable functions beyond the recursive functions, and no models of general computation that are not Turing-equivalent to TMs. [15] Nor would more computable functions be welcome to a community with a strong, close to universal, view that the class of recursive functions is already too large to model practical computation. (2) The role of feasible computation usually falls to P, the class of functions whose best sequential deterministic computations can be bounded in the number of steps by polynomials in the length of their inputs. On the other hand, Shor's quantum factoring algorithm [40] proves that substantial extension of P may be realistically feasible, depending on advances in quantum hardware.

[14] This may seem to be contradicted by [48, Section 9], but this claim concerns the original problem of [48, 9.1] and 3.1 below. Either the take-the-problem-as-solution approach of [48, Section 9]; [52, Section 4], and Sections 3–5 below, or some new, unexpected basis for algorithms that would satisfy Tarski's doubts—both constitute resolutions by extension for the first type of impasse. The other cases and examples listed in [48, 9.2–4], 3.2–4, and Section 4 below are mathematical results that illustrate delamination of the Tarski problem as far as it confronts CTT, but they do not resolve prior impasses themselves.

[15] The reader, however, will find elsewhere in this volume just such models; some are decades old, such as the inductive TMs of Burgin [6]. A new example is the work of Kieu [29] on a quantum solver for diophantine equations. Also see the series [8] edited by Calude et al.

(3) The computer scientists who put the practical ceiling at P still hold NP (assuming $P \subset NP$) to be more computable than the complements class co-NP. So in the sense of this paper, NP becomes their extended touchstone for abstract computation. Results on parallel computing are discussed below. (4) Moving beyond current technology, logicians and mathematicians explore infinitary computations, [16] real number machines, [17] and models for relativistic computation. [18]

4. Six approaches to computability with lack of uniformity

Uniformity is a frequent phenomenon, but not ubiquitous; when it fails, as in pseudorecursive theories, one can often view that failure as a near success instead. It becomes an indicator of intermediate potential, a middle ground, a *tertium quid*. In the examples we have seen, a theory $T = \bigcup_{n \in N} T_n$ has a given target complexity, but all the T_n have some lower fixed complexity. Equally important, in each setting there are similar T in which some T_n has complexity equivalent to T. For example, classical embeddings of a TM in a finitely based equational theory will usually require only a handful or so of variables. For these, T_{11} is already likely to be equivalent to T, which may be recursive, or not. More generally, T and all T_n may have the same complexity, or there may be a number k such that the T_i $(i \leqslant k)$ have one complexity, and the T_j $(j > k)$ and T have a higher complexity—a complexity break. These then are the extrema for which our pseudorecursive theories and analogues introduced below hold the middle ground. To provide terminology for this, we take *mediate computability* to mean T has higher complexity than all the T_n. [19] The term does not mean there must be “mediate computers.” We now turn to concrete approaches to mediate computability at different complexities and for different computation models.

4.1. Pseudorecursive and decidable theories (extension)

This solves the problem in 3.1 of a required Tarski procedure by assuming the problem can itself be abstracted as the answer and studying the consequences of this commitment. In other words, we propose pseudorecursiveness not as a classification of pathologies (although it can work that way), but as an extension of computability pseudorecursiveness for finitely based equational theories is itself the new notion of decidability. As mentioned earlier, this is a hallowed pattern in mathematics.

Here, the adoption of problem as solution is unsettling, for it reduces the Tarski question to a definitional tautology. This is not a satisfying conclusion even if it turns

[16] Hamkins et al. [20] extend the time resource for TMs from the natural to the ordinal numbers.

[17] Blum et al. [4] extend both decidability and P/NP to real and complex numbers.

[18] Hogarth [22,23] and Etesi et al. [16] discuss relativistic TMs and decision methods.

[19] It is possible that the complexities of the T_n have that of T as a limit. Examples are easily constructed by taking $X = \bigcup X_n$, with the complexities of the X_n below but approaching that of X, and then encoding X_n with n variables in the manner of the proof of Theorem 4.6 below. Such examples are beyond the scope of the current article.

out to be formally useful. More curiously, it has the consequence of creating hypercomputation by definition. This is mentioned in [50] and discussed further in Section 5 below. [20]

The examples below will mitigate the offense by showing reasonable and rigorous cases of mediate computability in other contexts—some smaller and feasible, others grander and fantastic. Although not discussed further here, for each case there are similar T with the complexity break described above. In other words, mediate computability is a proper notion in each setting.

4.2. Polynomial time vs. NPC and NC (specialization)

The first result concerns a theory that is nondeterministic-polynomial-time complete (NPC). Let W be an NPC subset of N. The constructions that led to the finite pseudorecursive base $\Psi 1$ (see Section 1 above) can be applied to W instead of the nonrecursive r.e. set X. The resulting equational theory will be P-reducible to W. Together with Theorem 1.2, this yields the following example.

Theorem 4.1 (Wells [51] Theorem 4.1). *There is a finite base Ψ_{NPC} for an equational theory of semigroups with additional unary operations and distinguished elements such that* $\mathrm{Th}(\Psi_{\mathrm{NPC}}) \in \mathrm{NPC}$, *but for all* $n \in N$, $\mathrm{Th}_n(\Psi_{\mathrm{NPC}}) \in \mathrm{P}$.

In Theorem 4.1, the set of NPC problems can be replaced with other subrecursive complexities higher than P, such as exponential time, NP space, superexponential time, etc. Moreover, we can go below P, if $\mathrm{NC} \subset \mathrm{P}$ (NC problems are decidable in polylogarithmic time on a polynomial number of parallel processors; see [19] for this and other terminology).

Theorem 4.2. *There is a finite base Ψ_{PC} for an equational theory of semigroups with additional unary operations and distinguished elements such that* $\mathrm{Th}(\Psi_{\mathrm{PC}})$ *is* P-*complete*, *but for all* $n \in N$, $\mathrm{Th}_n(\Psi_{\mathrm{PC}}) \in \mathrm{NC}$.

Proof. In fact, in Theorem 1.1, for any r.e. set $X \subset N$, $\mathrm{Th}(\Psi 1_X)$ is NC-equivalent to X, and all of the n-variable layers $\mathrm{Th}_n(\Psi 1_X)$ are in NC with bounded exponents, because fan-in is constant for the underlying TM and the polynomial bounds in the proof of Theorem 1.2 are all cubic. Now choose a P-complete problem Q and let $\Psi_{\mathrm{PC}} = \Psi 1_Q$. The resulting theory $\mathrm{Th}(\Psi_{\mathrm{PC}})$ will be NC-equivalent to Q, while the layers $\mathrm{Th}_n(\Psi_{\mathrm{PC}})$ will be in NC. □

4.3. K-sequential vs. K-parallel computing (alteration)

This approach concerns a novel distinction between sequential and parallel computing; see [51].

[20] Note that even if we reject this easy extension and its consequence of hypercomputation, finitely based pseudorecursive theories still represent mediate computability among r.e. equational theories.

Definition 4.3. By a *K-sequential* computation we shall mean a TM operating in Kalmár elementary time. Ackermann's function $A(m,n,p)$—recursive but not primitive recursive—may be characterized by

$$\begin{aligned} A(m,n,0) &= m+n, \\ A(m,n+1,p+1) &= A(m,A(m,n,p+1),p), \\ &\quad (m,n,p \geqslant 0), \end{aligned}$$

with two special cases

$$\begin{aligned} A(m,0,1) &= 0, \\ A(m,0,p) &= 1, \\ &\quad (m \geqslant 0, p \geqslant 2). \end{aligned}$$

Ackermann's diagonal function $\mathrm{Acd}(n) = A(n,n,n)$ is obviously recursive but also fails to be primitive recursive. A function (or set) is *K-parallelizable* iff it can be computed in Kalmár elementary time T by TMs operating simultaneously, sequentially, but not necessarily independently, where the number of machines grows as $\mathrm{Acd}(T)$. Acd and A are K-parallelizable and not K-sequential.

Let NSP be the class of K-parallelizable problems that are not K-sequential, and let Ksq be the class of K-sequential problems.

Theorem 4.4 (Wells [51] Theorem 4.4). *There is a finite base Ψ_{NSP} for an equational theory of semigroups with additional unary operations and distinguished elements such that* $\mathrm{Th}(\Psi_{\mathrm{NSP}}) \in \mathrm{NSP}$, *but for all* $n \in N$, $\mathrm{Th}_n(\Psi_{\mathrm{NSP}}) \in \mathrm{Ksq}$.

4.4. Higher degrees and enhanced TMs (relativization and alteration)

If we forego the finite equational bases of previous constructions, then there are pseudorecursive theories of every degree of unsolvability. Of course, mediate computability now depends on oracular TMs. These theories will exceed the power of any fixed oracular machine.

Definition 4.5. A *square-zero* semigroup satisfies $xx \approx yy \approx zxx \approx xxz$; the square of every element is zero. In these, we can take yy as the zero term. Tarski has pointed out that these equations are equivalent to the single equation $xx \approx yyy$. A unary operation h on such a semigroup is *nilpotent* iff $h(h(x)) \approx yy$ holds.

Theorem 4.6 (Wells [47] Theorem 10.7 and Corollary 10.8). *Let Y be any nonrecursive subset of N. There exists an infinite base $\Psi 2$ for a pseudorecursive variety of commutative square-zero semigroups with countably many additional nilpotent unary operations, such that $\Psi 2 \equiv_1 Y$ (or if Y is r.e., then $\Psi 2$ can be a recursive base) and* $\mathrm{Th}(\Psi 2) \equiv_{\mathrm{btt}} Y$. *In addition, all word problems for the variety are solvable but not uniformly solvable.*

We can combine the techniques used for constructing $\Psi 1$ of Theorem 1.1 and $\Psi 2$ to achieve finite types. Recall that a variety is *locally finite* iff all its finitely generated algebras are finite.

Theorem 4.7 (Wells [47] Theorem 10.9). *Let Y be any nonrecursive subset of N. There exists an infinite base $\Psi 3$ for a locally finite pseudorecursive variety of expanded square-zero semigroups with one unary nilpotent operation and two distinguished elements such that $\Psi 3 \equiv_1 Y$ (better, if Y is r.e., then $\Psi 3$ can be recursive) and* $\mathrm{Th}(\Psi 3) \equiv_1 Y$. $\mathrm{Th}(\Psi 3)$ *also has the property that all word problems for the variety are solvable but not uniformly solvable.*

4.5. Ordinal time and inductive TMs (relativization and alteration)

The pattern of the last example extends to ordinal-time TMs [20] and *m*-recursive structured-memory inductive TMs [7] in the following way.

For any ordinal α, let $L^{\mathrm{ord}}_{\alpha}$ be the class of languages $A \subseteq N$ (encoded as binary strings, say) such that there is an ordinal-time TM that decides A in α time, but no ordinal-time TM decides A in less than α time.

For every $k \in N$, let L^{ind}_{k} be the class of languages $A \subseteq N$ such that there is a k-recursive inductive TM that decides A, but no j-recursive inductive TM with $j<k$ decides A. By [7], L^{ind}_{m} is nonempty for infinitely many $m \in N$. Moreover, we know from [7] that m-recursive inductive TMs can decide all languages in $\Sigma_m \cup \prod_m$. We also observe that $L^{\mathrm{ind}}_{m} \cap \Sigma_{m+1} \subseteq \prod_{m+1}$ by induction, and this means $L^{\mathrm{ind}}_{m} \supseteq \Sigma_m - \prod_m$, which is nonempty for all $m \in N$.

Theorem 4.8. *For all limit ordinals γ, such that $L^{\mathrm{ord}}_{\gamma}$ is nonempty, and for all $m \in N$, there are equational bases $\Psi^{\mathrm{ord}}_{\gamma}$ and Ψ^{ind}_{m} for varieties of commutative square-zero semigroups with countably many additional nilpotent unary operations such that*

$$\mathrm{Th}(\Psi^{\mathrm{ord}}_{\gamma}) \in L^{\mathrm{ord}}_{\gamma},$$

$$\mathrm{Th}(\Psi^{\mathrm{ind}}_{m}) \in L^{\mathrm{ind}}_{m},$$

but for all $n \in N$, $\mathrm{Th}_n(\Psi^{\mathrm{ord}}_{\gamma})$ *and* $\mathrm{Th}_n(\Psi^{\mathrm{ind}}_{m})$ *are recursive—in fact, they are in* P.

Proof. Let Y be an arbitrary language in one of the L sets. The first conclusions are corollaries of the construction in Theorem 4.6 because the coding of the set Y in the equations in $\Psi 2$ is so direct. Because Theorem 1.2 covers Theorems 4.6–7, the n-variable layers are in P. □

4.6. Real computation (alteration)

As noted in Example 2.11 in [51], membership in the Mandelbrot set M can be characterized by using the iterated function based on $f_c(z) = z^2 + c$ for $c \in \mathrm{C}$. This iteration is commonly presented as a chain of equations; whether c lies in M is determined by the iteration on 0. Let $z_0(c) = z_0 = 0$; then

$$\begin{aligned} z_1 &= z_0^2 + c, \\ z_2 &= z_1^2 + c, \\ z_3 &= z_2^2 + c, \\ &\ldots \\ z_{i+1} &= z_i^2 + c, \\ &\cdots . \end{aligned}$$

If the $|z_i|$ are bounded—that is, if $\{z_i : i \in N\}$ is included in a disk of radius r centered at the origin ($r = 4$ suffices)—then $c \in M$. Using constant-length expressions and restricting the number of variables available is equivalent to an escape-time algorithm for charting M. Although these equations are definitions, not identities,[21] we see that if finitely many z_i are used, the result is decidable in the sense of Blum et al. [4], but if all the z_i are available, the result is undecidable.

We can now improve the analogy with examples of mediate computability if we consider the complement of M, which is semidecidable only.

Theorem 4.9. *For $n \in N$, let*

$$B_n = \{c \in \mathrm{C}\colon\ |z_n(c)| > 4\},$$

a decidable set defined using $n+1$ unary operation symbols $z_0, z_1, \ldots, z_n$. Then

$$\mathrm{C} - M = \bigcup_{n \in N} B_n$$

is a union of decidable sets that fails to be decidable; *i.e.*, *the union is not uniformly decidable*.

Here is a rough parallel with Theorem 4.1 for P_R and NPC_R where R is a ring and we use unit cost (see [4] for notation and definitions). Instead of universal equations or laws, we consider the existential problem: finding roots. But the number of variables remains the parameter of interest. For $n \in N$, with some abuse of notation, let

$$\mathrm{HN}_n/R = \{S : S \text{ is a finite set of polynomials with coefficients in } R \text{ and variables } v_0, \ldots, v_{n-1} \text{ that have a simultaneous zero over } R\},$$

$$\text{4-FEAS}_n/R = \{f : f \text{ is a degree-4 polynomial with coefficients in } R \text{ and variables } v_0, \ldots, v_{n-1} \text{ that has a zero over } R\},$$

[21] In fact, if the z_i are represented by unary function symbols and all $c \in \mathrm{C}$ are individual constants, then the definitions are identities. Or we could use individual constants z_i^c ($i \in N, c \in \mathrm{C}$). With unary symbols for norm and negation, we can express $|a| \leqslant 4$ as $4 + -|a| = |4 + -|a||$. Despite a form of mediate computability given in the next result, we find no pseudorecursive theory here, for the number of variables is irrelevant.

$$\mathrm{HN}/R = \bigcup_{n \in N} \mathrm{HN}_n/R,$$

$$\text{4-FEAS}/R = \bigcup_{n \in N} \text{4-FEAS}_n/R.$$

Theorem 4.10. *Let the ring R be one of* $\mathrm{Z}^<$ (*integers with order*), $\mathrm{R}^<$ (*real numbers with order*), *or* C (*complex numbers*). *For all* $n \in N$,

$$\mathrm{HN}_n/R,\ \text{4-FEAS}_n/R \in \mathrm{P}_R,$$

but

$$\mathrm{HN}/R \in \mathrm{NPC}_R, \quad \text{4-FEAS}/\mathrm{Z}^< \in \mathrm{NPC}_{\mathrm{Z}^<}, \quad \text{4-FEAS}/\mathrm{R}^< \in \mathrm{NPC}_{\mathrm{R}^<}.$$

Proof. Indeed, the P results are obvious. See Proposition 4.3 and Theorem 4.1 in [4] for the NP-complete results. It is important to note that 4-FEAS/$\mathrm{Z}^<$ (and so HN/$\mathrm{Z}^<$) is not decidable by the recursive unsolvability of diophantine equations (Hilbert's 10th problem [32]); thus, $\mathrm{P}_{\mathrm{Z}^<} \neq \mathrm{NP}_{\mathrm{Z}^<}$. On the other hand, HN/$\mathrm{R}^<$ (and so 4-FEAS/$\mathrm{Z}^<$) and HN/C are decidable by Tarski's results [42]; according to [4], the questions $\mathrm{P}_{\mathrm{R}^<} \neq \mathrm{NP}_{\mathrm{R}^<}$? and $\mathrm{P}_\mathrm{C} \neq \mathrm{NP}_\mathrm{C}$? are open. □

It is likely that there are more direct analogues of pseudorecursive equational theories that have similar mediate real and complex computability.

5. Hypercomputation by fiat; conclusions

Among the approaches to mediate computability given in Section 4, only 4.1 results in hypercomputation—and then only if we understand decidability to be implemented by some mechanizable decision procedure. Questions arise: what are the new machines? what are their computations in deciding these newly decidable problems? We could rely on previously mentioned models, such as Burgin's inductive TMs [6], or infinitistic machines, or oracular machines. We could rely on the graded strength of some of these models to match degree hierarchies of pseudorecursive theories. But these are easy answers that do not explicate or justify assigning computability as a special attribute of finitely based pseudorecursive theories. In short, we will not have new computer models until we have distinctive properties of the newly decidable theories.

What can distinguish finitely based pseudorecursive theories as decision problems? Why might they be more decidable than other r.e. nonrecursive theories, problems, sets, or relations? Why not extend a similar claim of decidability to at least any r.e. equational theory that is the union of an infinite, not uniform family of infinite nested (or disjoint) recursive subsets (and every r.e. theory, language, and set is [22])? If we

[22] The easiest decomposition is to choose a nest of increasingly larger finite subsets whose union is the original set. By fiat, every finite set is recursive. One may argue that this decree is no less artificial than declaring a pseudorecursive theory to be computable.

instead restrict ourselves to just a few theories, or to a single one, how do we make a natural or useful choice? When presented with these questions (and more), Tarski claimed that the number of variables that indexes the family T_n for a finitely based pseudorecursive theory T is an innocuous parameter that informally supports a uniform decision procedure for T. Formalizing this and understanding it precisely may be equivalent (see [48]).

The degree results of [47] show that pseudorecursive theories as "flat" sets are in fact not special from the point of view of classical recursive function theory. In other words, if we admit all finitely based pseudorecursive theories as computable sets of strings, then we have to accept all r.e. sets as computable. Thus we invite TMs with halting-problem oracles. We conclude that more structure is required to keep these newly decidable theories special. We can attempt an abstracted characterization that will preserve the recursive properties of the theories and some of their structure in order to separate the results from ordinary r.e. sets. An initial effort is described in [52].

So what do we gain by this decreed hypercomputation? In order to pursue the problem-as-solution method, we first need to know the properties expected to be invariant or at least desirable on moving to the larger domain of decidability. For example, in introducing rational numbers, such as $\frac{3}{5}$ to solve $3 \div 5$, we desire multiplication to distribute over addition.

Favored behavior could be derived from properties of the varieties such as operations, composition, and closure. Some general algebraic results are included in [47] and others will appear, but so far they offer no light on this issue.

In another direction, we could lift the machine properties embedded in the constructed equational bases (see Theorem 1.1) to adduce or induce detailed inherited mechanical structure at the decision problem level. This largely artificial bootstrapping threatens to be either trivial or circular.

A third path is to look at other logical theories and sets of sentences. More distant patterns lacking uniformity may be inspiring. For instance, the constants used at Theorem 4.9 seem as innocuous as variables, and the existential sentences of Theorem 4.10 treat variables equitably. Properties from other logical situations such as these may suggest useful invariants.

Finally, satisfying the quest for a naturally occurring pseudorecursive variety—one motivated by math or computing—might also broaden our appreciation and enthusiasm for the kind of structures and problems declared decidable. The notion of mediate computability might itself become more natural.

So far, there has emerged no concrete extension of computing models corresponding to the extension of decidability to finitely based pseudorecursive theories. Other examples in Section 4 do demonstrate considerable scope for usable, nontrivial (if unnatural) failures of uniformity that could be taken as a kind of "decidability" between the extrema native to each of those domains (and more). But these results on mediate computability also fail to inspire new machines.

An impasse has progressed but has not resolved. It is plausible that Tarski meant something significant by his declaration. The ontological claim that middle ground thereby exists between recursive and r.e. sets—between TMs and oracular TMs—is less compelling. I do not think the current impasse is what Tarski meant. But it is a place to start.

Postscript

Many impasses are called gaps. Gaps appear in math, in programs, in memories. Stub-outs, gaffes in print, bugs, blocks, holes in proofs, blanks, GPFs—all are places for mysterious, even Divine action. If gaps are closed, things get more reliable, more complete, better. Or do they? Although we need to be careful in formal or rigorous work, intuition may require and thrive on incompleteness. The London Underground exhorts, "Mind the gap." "Mine the gap" is also good advice.

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References

[1] L. Allison, A Practical Introduction to Denotational Semantics, Cambridge University Press, New York, 1986.
[2] S.N. Artemov, Explicit provability and constructive semantics, Bull. Symbolic Logic 7 (2001) 1–36.
[3] J. Barwise, J. Etchemendy, The Liar: An Essay on Truth and Circularity, Oxford University Press, New York, 1987.
[4] L. Blum, F. Cucker, M. Shub, S. Smale, Complexity and Real Computation, Springer, New York, 1998.
[5] L. Blum, M. Shub, S. Smale, On a theory of computation and complexity over the real numbers: NP-completeness, recursive functions and universal machines, Bull. Amer. Math. Soc. 21 (1989) 1–46.
[6] M.S. Burgin, Inductive Turing machines, Dokl. Akad. Nauk SSSR 270 (6) (1983) 1289–1293 (Russian) (Engl. Transl.: Soviet Math. Dokl. 27(3) (1983) 730–734).
[7] M.S. Burgin, The arithmetical hierarchy and inductive Turing machines, Dokl. Akad. Nauk SSSR 299 (3) (1988) 530–533 (Russian) (Engl. Transl.: Soviet Math. Dokl. 37(2) (1988) 390–393).
[8] C.S. Calude, J. Casti, M.J. Dinneen (Eds.), Unconventional Models of Computation; Proc. First Internat. Conf. on UMC, 1998, Springer, Berlin, 1998;
I. Antoniou, C.S. Calude, M.J. Dinneen (Eds.), Unconventional Models of Computation, UMC'2K; Proc. 2nd Internat. Conf. on UMC, 2000, Springer, Berlin, 2001;
C.S. Calude, M.J. Dinneen, F. Peper (Eds.), Unconventional Models of Computation; Proc. 3rd Internat. Conf. on UMC, 2002, Springer, Berlin, 2002.
[9] C.E. Cleland, Effective procedures and computable functions, Minds Mach. 5 (1995) 9–23.
[10] C.E. Cleland, Recipes, algorithms, and programs, Minds Mach. 11 (2001) 219–237.

[11] B.J. Copeland, D. Proudfoot, Alan Turing's forgotten ideas in computer science, Sci. Amer. 280 (1999) 99–103.
[12] J.H. Davenport, On the Integration of Algebraic Functions, Springer, New York, 1981.
[13] M. Davis, H. Putnam, J. Robinson, The decision problem for exponential diophantine equations, Ann. Math. 74 (2) (1961) 425–436.
[14] A. Ehrenfeucht, J.R. Geiser, C.E. Gordon, D.H.J. deJongh, A semantics for non-iterated local observation I, preprint, 1971.
[15] A.S. Esenin-Volpin, Le programme ultraintuitioniste des fondements des Mathématiques, in: Infinitistic Methods: Proc. Symp. Found. Math., Warsaw, 1959, Pergamon Press, Oxford, 1961, pp. 201–233.
[16] G. Etesi, I. Németi, Non-Turing computations via Malament–Hogarth space-times, Internat. J. Theoret. Phys. 41 (2002) 341–370, available in electronic form from http://arXiv.org/pdf/gr-qc/0104023.
[17] K.O. Geddes, S.R. Czapor, G. Labahn, Algorithms for Computer Algebra, Kluwer Academic, Boston, 1992.
[18] J.R. Geiser, A formalization of Essenin–Volpin's proof theoretical studies by means of nonstandard analysis, J. Symbolic Logic 39 (1974) 81–87.
[19] R. Greenlaw, H.J. Hoover, W.L. Ruzzo, Limits to Parallel Computation: P-Completeness Theory, Oxford University Press, New York, 1995.
[20] J.D. Hamkins, A. Lewis, Infinite time Turing machines, J. Symbolic Logic 65 (2000) 567–604.
[21] J. Henle, E. Kleinberg, Infinitesimal Calculus, MIT Press, Cambridge, MA, 1979.
[22] M.L. Hogarth, Non-Turing computers and non-Turing computability, in: D. Hull, M. Forbes, R.M. Burian (Eds.), PSA 1, Philosophy of Science Association, East Lansing, MI, 1994, pp. 126–138, available in electronic form from http://www.hypercomputation.net/resources.html.
[23] M.L. Hogarth, Deciding arithmetic in Malament–Hogarth spacetimes, British J. Philos. Sci., accepted for publication, 2002; available in electronic form from http://www.hypercomputation.net/download/2002a_hogarth.pdf.
[24] M.R. Holmes, Strong axioms of infinity in NFU, J. Symbolic Logic 66 (2001) 87–116.
[25] D. Isles, What evidence is there that 2^{65536} is a natural number? Notre Dame J. Formal Logic 33 (1992) 465–480.
[26] T.J. Jech, Set Theory, 2nd Edition, Springer, New York, 1997.
[27] C.G. Jockusch Jr., review of R.A. Shore, T.A. Slaman, Defining the Turing jump, Bull. Symbolic Logic 7 (2001) 73–75.
[28] A. Jung (Ed.), Domains and denotational semantics: history, accomplishments, and open problems, Bull. EATCS 59 (1996) 227–256.
[29] T.D. Kieu, Numerical simulations of a quantum algorithm for Hilbert's tenth problem, available in electronic form from http://arxiv.org/PS_cache/quant-ph/pdf/0304/0304114.pdf.
[30] K. Kuratowski, A. Mostowski, Set Theory, North-Holland, Amsterdam, 1967.
[31] A.I. Mal'cev, The Metamathematics of Algebraic Systems—Collected Papers: 1936–1967 (newly translated, edited, and annotated by Benjamin Wells), North-Holland, Amsterdam, 1971.
[32] Y.V. Matiyasevich, Hilbert's Tenth Problem, MIT Press, Cambridge, MA, 1993.
[33] S. Mazur, Computable Analysis, as Rozprawy Matematyczne IMPAN (Dissertationes Mathematicae) Vol. 33, Państwowe Wydawnictwo Naukowy, Warsaw, 1963.
[34] V. McGee, Review of papers in set theory by George Boolos, Bull. Symbolic Logic 7 (2001) 60–62.
[35] J. Moses, Symbolic integration: the stormy decade, Comm. ACM 14 (1971) 548–560.
[36] H.-O. Peitgen, H. Jürgens, D. Saupe, Chaos and Fractals: New Frontiers of Science, Springer, New York, 1992.
[37] H. Putnam, Seminar notes on Kalmár's proof of Gentzen's Theorem, MIT, Cambridge, MA, 1962.
[38] R.H. Risch, The problem of integration in finite terms, Trans. Amer. Math. Soc. 139 (1969) 167–189.
[39] A. Robinson, Complete Theories, North-Holland, Amsterdam, 1956.
[40] P.W. Shor, Algorithms for quantum computation: discrete logarithms and factoring, in: S. Goldwasser (Ed.), Proc. 35th Annu. Symp. Foundations of Computer Science, IEEE Computer Society Press, Los Alamitos, CA, 1994, pp. 124–134.
[41] I. Stewart, The Lorenz attractor exists, Nature 406 (2000) 948.
[42] A. Tarski, A Decision Method for Elementary Algebra and Geometry, 2nd Edition, University of California, Berkeley, 1951.

[43] A. Tarski, Logic, Semantics, Metamathematics: Papers from 1923 to 1938, 2nd Edition, Hackett, Indianapolis, 1983.
[44] W. Tucker, The Lorenz attractor exists, Ph.D. Thesis, Vol. 11, Uppsala Dissertations in Mathematics, 1998.
[45] B. Wells, Pseudorecursive varieties and their implications for word problems, Doctoral Dissertation, University of California, Berkeley, 1983.
[46] B. Wells, Pseudorecursiveness: implications for decidability and computability, J. Symbolic Logic 51 (1986) 858–859.
[47] B. Wells, Pseudorecursive varieties of semigroups—I, Int. J. Algebra Comput. 6 (1996) 457–510.
[48] B. Wells, Is there a nonrecursive decidable equational theory? Minds Mach. 12 (2002) 303–326 (special issue on effective procedures, edited by C.E. Cleland).
[49] B. Wells, Pseudorecursive varieties of semigroups—II, Internat. J. Algebra Comput., to appear.
[50] B. Wells, Hypercomputation by definition, Abs. Amer. Math. Soc. 24 (2003) 987-03-14.
[51] B. Wells, Applying, extending, and specializing pseudorecursiveness, issue dedicated to the Alfred Tarski Centenary Conference, Warsaw, May 2001, Ann. Pure Appl. Logic 126 (2004) 225–254.
[52] B. Wells, Abstracting and generalizing pseudorecursiveness, Algebra Universalis, special issue on the Internat. Conf. on Modern Algebra Honoring the 60th Birthday of Ralph McKenzie, May 2002, submitted for publication.
[53] Wolfram Research, Inc., http://integrals.wolfram.com/about/history/.
[54] A.S. Yessenin-Volpin, The ultra-intuitionistic criticism and the antitraditional program for foundations of mathematics, in: A. Kino, J. Myhill, R.E. Vesley (Eds.), Intuitionism and Proof Theory, Proc. Summer Conf. at Buffalo, NY, 1968, North-Holland, Amsterdam, 1970, pp. 3–52.