Remarks on a 25 year old Theorem on Two-dimensional Cellular Automata

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We state and prove a theorem which characterizes those initial patterns which persist for all time in a certain two-dimensional cellular automaton modelling an excitable medium. The proof is based on a winding number concept introduced in a previous paper.

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The purpose of this brief note is to call attention to a theorem published almost twenty-five years ago [2] on two dimensional cellular automata which we believe is still of interest, but which seems to be very little known. One reason for this obscurity may be that when we wrote this paper, we did not know the term “cellular automata”, and so it does not appear in the title or elsewhere in the paper. A second reason may be that the journal in which it appeared was discontinued more than 15 years ago. Despite the long intervening period, we are not aware of another theorem like it in this field.

The cellular automaton (CA) in question is quite well known, as a discrete model of “excitable media”. See for example [3], where the CA is described (with acknowledgement) and some of the spiral-like patterns it supports are shown. What seems to be largely unknown is that we proved a theorem which allows one to predict from the initial condition an

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important aspect of the long time behavior of this model. The theorem is proved by showing that a certain “topological invariant” remains constant as the system evolves.

In this note we give a result which follows from the main theorem in [2] and which is more easily described than that theorem. This result was stated informally in [1], but without proof, so we prove it here, starting from the conclusions of [2].

We consider a specific three-state CA on a two-dimensional square lattice. This means that to each index pair \((i, j)\), and for each integer valued “time” \(n \geq 0\), we associate a number from the set \(\{0, 1, 2\}\), denoted by \(s_n(i, j)\), and this mapping obeys a set of rules which enables us to determine \(s_{n+1}(i, j)\) if we know \(s_n(i, j)\) and also the values of \(\{s_n(i', j')\}\) for \((i', j')\) lying in some “neighborhood set” of \((i, j)\). The nature of this neighborhood does not matter much in the statement of the theorem. Indeed, the result is not restricted to two dimensions. For the purpose of this basic example, consider that the neighborhood of the “cell” \((i, j)\) consists of the set \(N_{ij} = \{(i+1, j), (i-1, j), (i, j+1), (i, j-1)\}\). This is often called the “von Neumann” neighborhood of the cell \((i, j)\).

Our rules are as follows:

\[
s_{n+1}(i, j) = \begin{cases} 
2 & \text{if } s_n(i, j) = 1 \\
1 & \text{if } s_n(i, j) = 0 \text{ and } s_n(i', j') = 1 \text{ for at least one } (i', j') \in N_{ij} \\
0 & \text{otherwise}
\end{cases}
\]

The reader who has not seen this model before may wish to consider the following example. In this example we pick out for examination a square block of four cells which at time \(n = 0\) are in the states shown:

\[
\begin{array}{cc}
1 & 2 \\
0 & 0
\end{array}
\]

Use the rule to determine the states of these four squares at future times \(n = 1, 2, 3\). Actually, we have not given enough information to do this exactly. The states of three of these cells at time \(n = 1\) are completely determined by what we can see above, but the state of the lower right cell depends also on neighbors which we have not shown. Nevertheless, if we follow all possible alternatives for the arrangement above, we find only two possibilities:

(i) After one time step we have the same sub-pattern as before, except that it is rotated by 90 degrees,
or,

(ii) After three time steps we have returned to the original states in these four squares.

We quickly conclude that if one has the sub-pattern above to begin with, then for all \( n \geq 0 \), at least two of these four cells must have non-zero state. We call the overall pattern “persistent”, because within a fixed bounded region it never goes to all zeros.

Our theorem was more interesting than this. As restricted to this model it says:

**Theorem 1:** If the initial state space \( \{ s_0(i, j) \mid -\infty < i, j < \infty \} \) contains any copies, or rotations or reflections, of the sub-pattern above, or of one of the configurations

\[
\begin{array}{cccc}
1 & 2 & 1 & 1 \\
0 & 2' & 2 & 0
\end{array}
\]

then the pattern is persistent. If none of these sub-patterns exists at \( t = 0 \), and the number of non-zero cells initially is finite, then the pattern is not persistent.

We have emphasized the interesting part of the theorem.

Above we referred to a “topological invariant” which we discovered for this model. This is used in the proof of the theorem. To define this invariant we consider any cycle of cells, which we denote by

\[ C = \{ C_1, C_2, \ldots, C_n \}, \]

where \( C_{j+1} \) is a neighbor of \( C_j \) and where \( C_n \) is a neighbor of \( C_1 \). We also consider that the states \( \{0, 1, 2\} \) are on a circle. At a certain time \( n \) we proceed from \( C_1 \) along the cycle of cells, and as we do so, we move from point to adjacent point on the circle. That is, at step \( k \) we are at state \( \sigma \in \{0, 1, 2\} \) if the state \( s_n(i_k, j_k) \) of cell \( C_k \) at time \( n \) is equal to \( \sigma \). When we return to \( C_1 \) we will have made some net integer number of clockwise rotations around the circle. We denote this number by \( W_n(C) \), and call it the “winding number” of \( C \). Our basic lemma is that \( W_n(C) \) is independent of \( n \).

The result proved in [2] is not quite Theorem 1 above. We showed there that a pattern is persistent if and only there is some cycle \( C \) such that
$W_0(C) \neq 0$. To obtain Theorem 1 we must show that if there is any cycle with non-zero winding number, then there is one with only four cells. (The configurations pictured above, plus their rotations and reflections, are all of the four cell sub-patterns with non-zero winding number.) Since this argument has not been given elsewhere, we present it here.

It is sufficient to show that if there is a cycle $C$ with $n$ cells where $n > 4$, and with non-zero winding number, then there is another cycle with fewer than $n$ cells and non-zero winding number.

The cycle $C = \{C_1, \ldots, C_n\}$ may or may not have a proper subcycle $\{C_j, \ldots, C_{j+k}\}$, $(k \geq 3)$, where $C_{j+k}$ is a neighbor of $C_j$. If there is such a subcycle and it has non-zero winding number, then we have done the necessary contraction. If there is a subcycle with zero winding number, then the cells $C_{j+1}, \ldots, C_{j+k-1}$ can be deleted from $C$ without changing the winding number of $C$.

Hence we only have to consider the case where $C$ has no proper subcycles at all. In this case we consider a cell $C_k = (i, j) \in C$ such that $i + j$ is as large as possible within $C$. Then we can assume that $C_{k-1} = (i-1, j)$ and $C_{k+1} = (i, j-1)$ or vice-versa. (Here we may need to identify $C_{n+1}$ with $C_1$.) Thus $C_{k-1}$, $C_k$, $C_{k+1}$ are three cells in a unique square of cells, and since $C$ has no subcycles, the fourth cell, say $C' = (i-1, j-1)$, is not in $C$. If the square cycle $\{C_{k-1}, C_k, C_{k+1}, C'\}$ has non-zero winding number then we are done. If this square cycle has zero winding number, then in $C$ we replace the upper right square $C_k$ with the lower left square $C'$, without changing $W_0(C)$. We have not reduced the number of cells in $C$, but we have reduced the size of $C$ in the sense that either the maximum $i + j$ is reduced, or there are fewer cells attaining this maximum. Continuing in this way we must reach one of the configurations considered before. This completes the proof of the Theorem.

REFERENCES

