1. (a) The initial value is $A_0 = 1000$. The recurrence for A_n is

$$A_n = 1.0x \cdot A_{n-1} - y.$$

(b) Using the iteration method from page 152 in the text (or page 15 in the notes) we get:

$$A_n = 1000 \cdot (1.0x)^n - \frac{(1.0x)^n - 1}{1.0x - 1} \cdot y.$$

Note that $1.0x - 1 \neq 0$ because x is at least 1.

2. The initial values are $t_0 = 1$ and $t_1 = 3$.

By a *correct* string we mean a ternary string that does not contain two consecutive 0s. Thus t_n is the number of correct strings of length n.

To get a recurrence for t_n we note the following. A correct strings of length n either (i) begins with a 0, (ii) begins with a 1 or (iii) begins with a 2.

(i) If a correct string of length n begins with a 0, the second symbol must be a 1 or 2 (because otherwise have two consecutive 0s) and the last $n-2$ symbols can be any correct string. Thus, the number of correct strings of length n that begin with 0 is $2 \cdot t_{n-2}$.

(ii) If a correct string of length n begins with a 1, the remaining $n-1$ symbols can be any correct strings of length $n-1$. Thus the number of correct strings of length n that begin with a 1 is t_{n-1}.

(iii) Similarly as in (ii), the number of correct strings that begin with a 2 is t_{n-1}.

Since the cases (i)–(iii) are disjoint, the value of t_n is the sum of the values from (i)–(iii) and we get the recurrence

$$t_n = 2 \cdot t_{n-1} + 2 \cdot t_{n-2}.$$

Using the recurrence (and the initial values) we calculate $t_2 = 8$, $t_3 = 22$, $t_4 = 60$ which then gives $t_5 = 164$.

3. The characteristic equation for the recurrence for b_n is

$$r^2 + 2r + 1 = 0.$$

The characteristic equation has only one repeated root $r = -1$. Therefore the recurrence has a solution of the form

$$b_n = \alpha_1(-1)^n + \alpha_2 \cdot n \cdot (-1)^n$$

To find the values of the constants α_1 and α_2 we use the initial values of b_n. We have

$$b_0 = \alpha_1 + \alpha_2 \cdot 0 = 5, \text{ and,}$$

$$b_1 = \alpha_1(-1) + \alpha_2 \cdot 1 \cdot (-1) = 1.$$

From the first equation we get $\alpha_1 = 5$ and substituting this value to the second equation we get $\alpha_2 = -6$. This gives the solution

$$b_n = 5(-1)^n - 6n(-1)^n.$$

4. The characteristic equation for the recurrence for a_n is

$$r^2 - 2r - 2 = 0.$$

The two distinct roots for the characteristic equation are $1 + \sqrt{3}$ and $1 - \sqrt{3}$. Therefore the recurrence relation has a solution of the form

$$a_n = \alpha_1(1 + \sqrt{3})^n + \alpha_2(1 - \sqrt{3})^n$$

To find the values of the constants α_1 and α_2 we use the initial values of a_n. We have

$$a_0 = \alpha_1(1 + \sqrt{3})^0 + \alpha_2(1 - \sqrt{3})^0 = \alpha_1 + \alpha_2 = 1, \text{ and,}$$

$$a_1 = \alpha_1(1 + \sqrt{3}) + \alpha_2(1 - \sqrt{3}) = 3.$$

Subtracting both sides of the first equation from the second equation yields:

$$(\alpha_1 - \alpha_2) \cdot \sqrt{3} = 2$$

Solving the last equation together with $\alpha_1 + \alpha_2 = 1$ yields:

$$\alpha_1 = \frac{1}{2} + \frac{1}{\sqrt{3}}, \quad \alpha_2 = \frac{1}{2} - \frac{1}{\sqrt{3}}.$$

A closed form expression for a_n is then

$$a_n = (\frac{1}{2} + \frac{1}{\sqrt{3}})(1 + \sqrt{3})^n + (\frac{1}{2} - \frac{1}{\sqrt{3}})(1 - \sqrt{3})^n$$

5. (a) The rows and columns in the adjacency matrices are ordered 1, 2, 3, 4, 5. The adjacency matrix for G_1 is

$$A_{G_1} = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

The adjacency matrix for G_2 is

$$A_{G_2} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

(b) The graphs G_1 and G_2 are not isomorphic. This follows from the observation that the vertex 2 of G_2 has degree four and G_1 has no vertex of degree four. A possible graph isomorphism has to preserve the degree of vertices.

(c) The graphs H_1 and H_2 are isomorphic. A possible graph isomorphism that maps H_1 to H_2 can be defined as

$$\varphi(2) = 5,\ \varphi(4) = 3,\ \varphi(3) = 2,\ \varphi(1) = 1,\ \varphi(5) = 4.$$

6. (a) The vertices e and c have odd degree. Since not all vertices of H_3 have even degree, H_3 has no Eulerian circuit.

(b) H_3 has two vertices of odd degree (e, c) and all other vertices have even degree. This means that H_3 has an Eulerian trail. One possible Eulerian trail is

$$(e, a, d, e, b, a, f, d, c, f, b, c)$$

(Note that the end points of the Eulerian trail must be the two vertices of odd degree.)

7. (a) H_4 does not have a Hamiltonian cycle. This can be justified as follows. Consider an arbitrary closed walk W of H_4 that visits all the vertices - W has to begin and end with the same vertex. The edge $\{c, f\}$ is a cut-edge and hence the edge $\{c, f\}$ would need to be traversed two times in the closed walk W. This means that one of the vertices c or f would need to occur more than once in W (where the two occurrences are not the begin and the end of the walk). This means that W cannot be a Hamiltonian cycle.

(b) One possible Hamiltonian path of H_4 is

$$(a, b, c, f, d, e)$$