INSTRUCTIONS

• You have 40 minutes. Attempt all four questions.

• You may bring in one 8.5 × 11 sheet of paper containing notes, and use it during the midterm.

• Answer each question in the space provided (on the question paper). There is an extra page at the end of the exam if more space is needed. Please write legibly.

• Note: In questions dealing with counting, combinatorics or probability it is not expected that you should compute large numerical values: it is fine to give the final answer in a form like \( \frac{56!}{29!} \) or \( \binom{40}{21} \) as long as you clearly explain how you arrived at the answer.

• Please note: You are asked to write your answers using a non-erasable pen. Answers written in pencil or erasable ink will be marked, but they will not be considered for remarking after the midterms are returned.

STUDENT NUMBER: One digit in each square, please!

Student number (written in words):

MARKS

| Problem 1 | /X |
| Problem 2 | /Y |
| Problem 3 | /Z |
| Problem 4 | /W |
| Total     | /X + Y + Z + W |

Code: aijkklkkhijjopewatwu2tuytuen1duaty4dtyulan2ypaqw
1. Consider relations $R \subseteq A \times B$ and $S \subseteq B \times C$. Prove that
\[(R \circ S)^{-1} = S^{-1} \circ R^{-1}.\]

Here $R \circ S$ denotes the composition of relations $R$ and $S$. 
2. A three-digit natural number is a number of the form $d_1d_2d_3$, where $d_1$, $d_2$, and $d_3$ are elements from the set $\{0, 1, \ldots, 9\}$ and $d_1 \neq 0$. In other words, a three-digit natural number is any number between 100 and 999.

(i) How many three-digit natural numbers can be formed if each digit must be distinct?

(ii) How many three-digit natural numbers can be formed if the number is even and digits may be repeated? (A number is even if the digit $d_3$ is an element from the subset $\{0, 2, 4, 6, 8\}$.)

(iii) How many three-digit natural numbers can be formed if the number is even, digits may be repeated, and the number contains the digit 5?
3. How many functions from the set \{1, 2, \ldots, n\} to \{0, 1\} \ (n \geq 1) there exist

(i) that are one-to-one?
(ii) that map both 1 and \(n\) to 0?
(iii) that map exactly one of the integers 1, \ldots, n - 1 to 1?

Justify your answers!

*Note:* In some of the cases one may need to consider separately a few small values of \(n\).
4. A standard deck of playing cards contains 52 cards. Each card has a rank from the set \{2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A\} and a suit from the set \{♣, ♢, ♥, ♠\}.

Assume that the ranks J, Q, and K have numerical values of 10 and assume that the rank A has a numerical value of 11.

For the following questions, let \(X\) be a random variable corresponding to the numerical value of the first card drawn from a shuffled deck and let \(Y\) be a random variable corresponding to the numerical value of the second card drawn from the same shuffled deck.

Below \(P(e)\) stands for probability of event \(e\) and \(E(X)\) is the expected value of random variable \(X\).

(i) What is \(P(X \text{ is even})\)?
(ii) What is \(E(X)\)?
(iii) Are \(X\) and \(Y\) independent? Explain why or why not.
(iv) Using your answer from part (ii), determine \(E(X + Y)\).