Sets, functions, relations (review)

set: unordered collection of objects

$\{3, 5, 7, 10, 14\}$

$= \{10, 7, 3, 14, 5\} = \{10, 7, 7, 3, 14, 5\}$

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$

$\mathbb{Q} = \{ \frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0 \}$

$A = \{ m \in \mathbb{Z} \mid m \bmod 5 = 0 \}$

cardinality of a finite set B: $|B|$

empty set $\emptyset$, $|\emptyset| = 0$

Operations

set inclusion $A \subseteq B$

strict inclusion $A \subsetneq B$; $A \subseteq B$ and $A \neq B$

Union: $A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$

Intersection: $A \cap B = \{ x \mid x \in A \text{ and } x \in B \}$

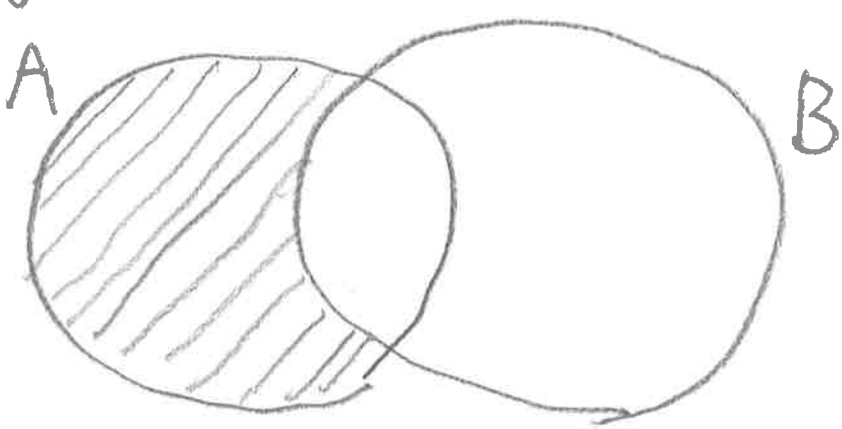
Set difference:

$$A - B = \{ x \in A \mid x \notin B \}$$

Complement operation w.r.t. a universe set U:

$$\overline{A} = U - A$$

Venn diagram for $A - B$:



Computation laws for operations:

Associativity

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

can write: $A_1 \cup A_2 \cup A_3 \cup A_4$

Distributivity

$$(*) \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

De Morgan's laws:

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

<u>Claim.</u> $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

<u>Proof.</u> To show equality of sets we need to show inclusion in both directions.

"$\subseteq$": Consider $x \in A \cap (B \cup C)$

This means: $x \in A$ and ($x \in B$ or $x \in C$)

Using distributivity of "and", and "or" we get:

($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$)

By definition of union and intersection we get

$$x \in (A \cap B) \cup (A \cap C)$$

"$\supseteq$" is left as an exercise

Cartesian product of sets

$$A \times B = \{ (a,b) \mid a \in A,\ b \in B \}$$

- set of ordered pairs
- extended in the natural way for k sets

Power set (set of subsets)

$$2^A = \{ B \mid B \subseteq A \}$$

- if A is finite

$$|2^A| = 2^{|A|}$$

Example. $A = \{1, 2, 3\}$

$$2^A = \{ \phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \}$$

Functions

- a function associates a unique "output" to each "input"

$f : A \to B$: gives a rule that assigns to each $a \in A$ a unique element $f(a) \in B$

- A is the domain of f
- B is the range
- the image of f : $\operatorname{im} f = \{ b \in B \mid \exists a \in A : f(a) = b \}$

Def. • f is one-to-one if
$(\forall a_1, a_2 \in A)\ f(a_1) = f(a_2)$ implies $a_1 = a_2$

• f is onto if
$(\forall b \in B)(\exists a \in A)\ f(a) = b$

• f is a bijection if it is one-to-one and onto

Composition of function $f: A \to B$, $g: B \to C$

$$(g \circ f): A \to C$$

$$(g \circ f)(a) = g(f(a)) \quad \forall a \in A$$

Note: Composition is associative:

$$(f \circ g) \circ h = f \circ (g \circ h)$$

Growth rate of functions – "big/small Oh" notations

$f, g : \mathbb{N} \to \mathbb{R}_+ \cup \{0\}$

$f(n) = O(g(n))$:

$\exists c, n_0 : f(n) \leq c \cdot g(n)$ for $n \geq n_0$

$f(n) = o(g(n))$:

$(\forall c > 0)(\exists n_c) : f(n) \leq c \cdot g(n)$ for $n \geq n_c$

Intuitive:

$f(n) = O(g(n))$: "f is upper bounded by g"

$f(n) = o(g(n))$: "f is asymptotically smaller than g"

Examples.

If $p(n)$ is a polynomial of rank r with positive coefficients, then $p(n) = O(n^r)$.

For $a < b$: $n^a = o(n^b)$

For $a, b > 0$: $\log_a n = O(\log_b n)$

For $c > 0$: $\log n = o(n^c)$

For any polynomial $p(n)$, $p(n) = o(2^n)$

$$7n^2 + 5n + 18 = O(n^2)$$

$$n \cdot \sqrt{n} = o(n^2)$$

Relations

Informally: a relationship that can exist between members of given sets

Often we consider binary relations:

relation from set A to set B is

$$R \subseteq A \times B$$

Examples of relations from A to A:

- $\emptyset$ empty relation
- identity relation

  $$I_A = \{(a, a) \mid a \in A\}$$

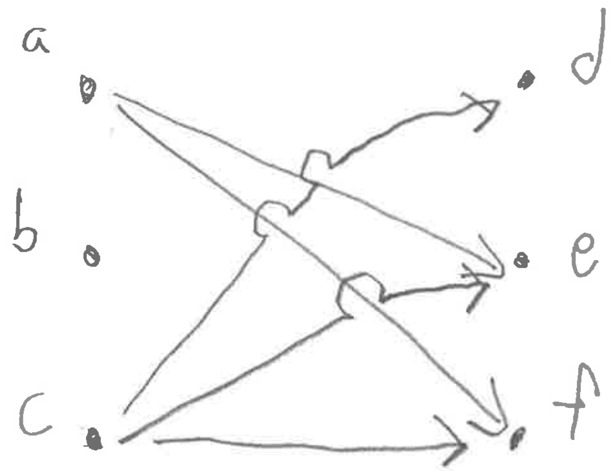
- universal relation $U_A = A \times A$

Example. $A = \{a, b, c\}$ $B = \{d, e, f\}$

$R \subseteq A \times B$

$R = \{(a,e), (a,f), (c,d), (c,e), c,f)\}$

Graph representation of R:



From last time:

binary relation from set A to set B:

$$R \subseteq A \times B$$

Extension: n-ary relation

$$R_n \subseteq A_1 \times A_2 \times \ldots \times A_n$$

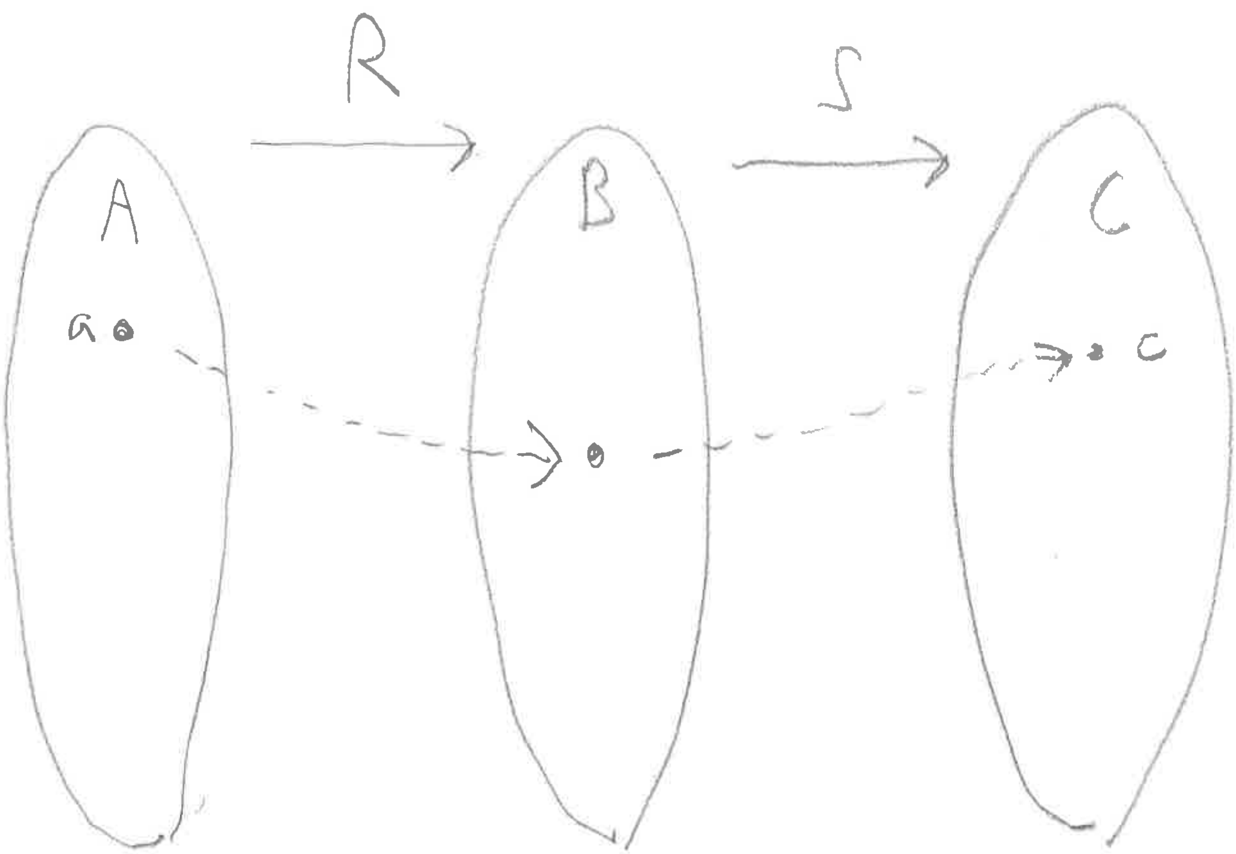
Most of the time we deal with binary relations.

Operations on (binary) relations:

Composition $R \subseteq A \times B$, $S \subseteq B \times C$

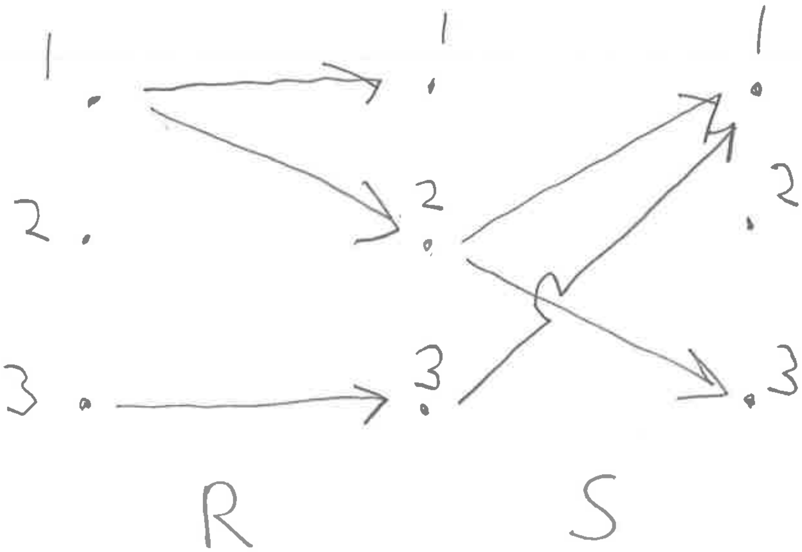
$$R \circ S = \{(a,c) \mid (\exists b \in B)\,(a,b) \in R,\ (b,c) \in S\}$$

$\subseteq A \times C$



$(a,c) \in R \circ S$

Example. $R, S \subseteq \{1,2,3\} \times \{1,2,3\}$

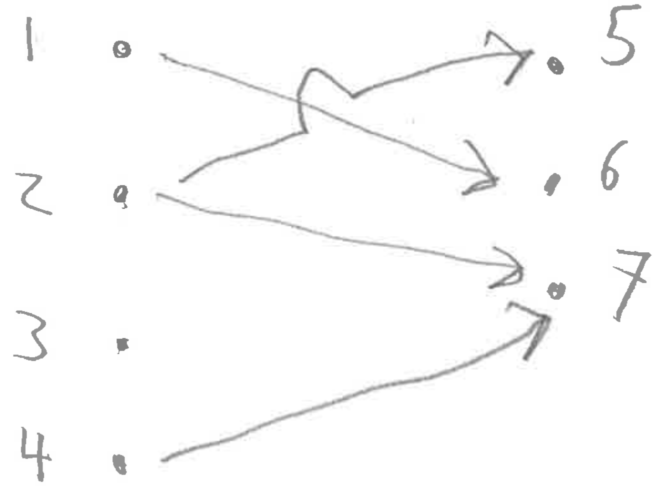


$$R \circ S = \{(1,1), (1,3), (3,1)\}$$

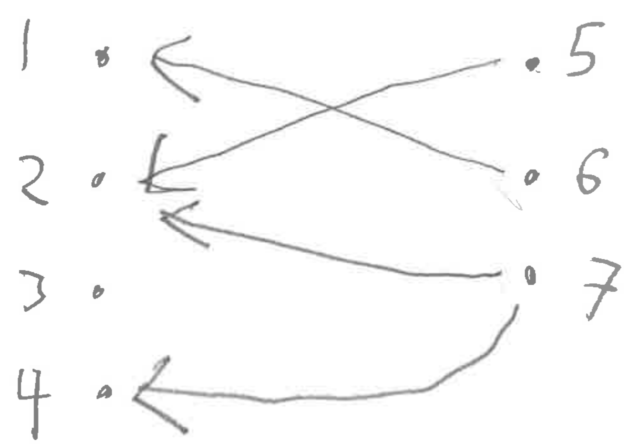
Inverse of a relation $R \subseteq A \times B$

$$R^{-1} = \{ (b, a) \mid (a, b) \in R \}$$

Ex. R



R^{-1}:



Graph representation of R^{-1} is obtained from representation of R by reversing the arrows.

Example. $R \subseteq A \times B$, $S \subseteq B \times C$

Claim: $(R \odot S)^{-1} = S^{-1} \odot R^{-1}$

Proof. Both $(R \odot S)^{-1}$ and $S^{-1} \odot R^{-1}$ are subsets of $C \times A$.

$(c, a) \in (R \odot S)^{-1}$ iff

$(a, c) \in R \odot S$ iff

$(\exists b \in B) : (a, b) \in R$ and $(b, c) \in S$ iff

$(\exists b \in B) : (b, a) \in R^{-1}$ and $(c, b) \in S^{-1}$ iff

$(c, a) \in S^{-1} \odot R^{-1}$

Note: the composition operations for functions and relations have slightly different definition.

- composition of relations $R \odot S$:
  apply first R and then S

- composition of functions $f \circ g$:
  apply first g and then apply f $\quad (f(g(x)))$

textbook definition

We use different symbols for function composition and relation composition.

Defining functions as <u>special case</u> of relations:

A function $f: A \to B$ is a relation $f \subseteq A \times B$ such that

(i) $(\forall a \in A)(\exists b \in B) \quad (a,b) \in f$ // f defined for all $a \in A$

(ii) $(\forall a \in A)(\forall b_1, b_2 \in B)$
$[(a,b_1) \in f \text{ and } (a,b_2) \in f]$ implies $b_1 = b_2$

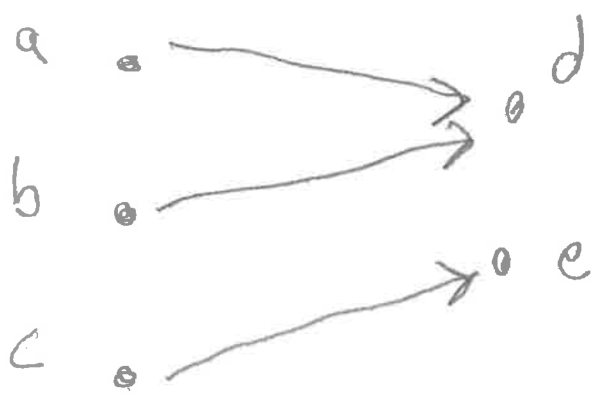
// all elements of A have a unique image

When a function $f: A \to B$ is viewed as a relation, the inverse relation f^{-1} is uniquely defined.

Note: f^{-1} need not be a function.

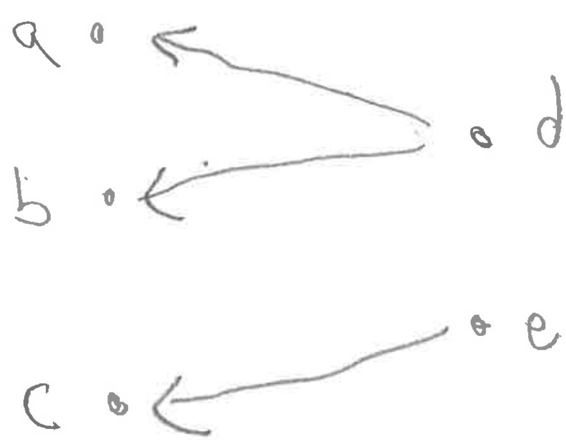
Example.

f



f^{-1} is not a function

inverse of the relation f



f^{-1} "maps" d to two elements

<u>Note</u>: $f: A \to B$. If f^{-1} is a function then it is the unique inverse of f. (but, in general, a function does not need to have an inverse).

<u>Definition</u>. Consider $f: A \to B$ and $g: B \to A$.

(i) g is a <u>left-inverse</u> of f if

$$g \circ f = 1_A \text{ (identity function on } A)$$

(ii) g is a <u>right inverse</u> of f if

$$f \circ g = 1_B$$

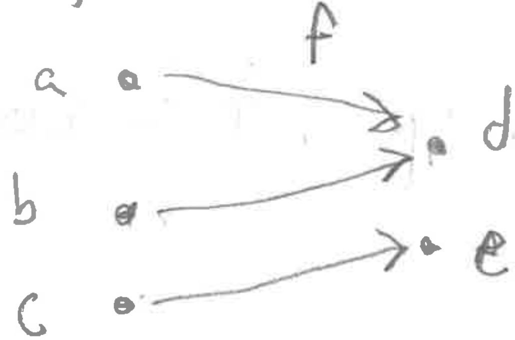
(iii) g is a <u>two-sided inverse</u> of f if

$$g \circ f = 1_A \text{ and } f \circ g = 1_B.$$

<u>Example.</u> Right inverse of a function need not be unique.

$A = \{a, b, c\}$ $B = \{d, e\}$

$f : A \to B$, $f(a) = f(b) = d$, $f(c) = e$



Right inverse g_1: $g_1(d) = a$
$g_1(e) = c$

Another right inverse g_2: $g_2(d) = b$
$g_2(e) = c$

$$f \circ g_1 = f \circ g_2 = 1_B$$