Def: A graph isomorphism between $G = (V, E)$ and $G' = (V', E')$ is a bijective mapping $\phi: V \rightarrow V'$ such that for all $u, v \in V$:

$$\{u, v\} \in E \iff \{\phi(u), \phi(v)\} \in E'$$

Note: An isomorphism between $G$ and $G'$ must preserve:

- number of vertices and number of edges
- degree of corresponding vertices
- cycle of given length
- etc.

If $G$ and $G'$ do not satisfy these properties, then they are not isomorphic.
Example.

G and H are not isomorphic because vertex 2 in H has degree four and G has no vertex of degree four.
Note: Consider $G = (V, E)$, $G' = (V', E')$. To show that $\phi: V \rightarrow V'$ is an isomorphism, we need to show that $\phi$ preserves the presence and absence of edges.

One helpful way to do this is to show that the adjacency matrix of $G$ is the same as the adjacency matrix of $G'$ when rows and columns are labeled in the "order determined by $\phi$".

Consider the following example.
Example.

$G$: 

$G'$:

Isomorphism $\phi$: 

$u_1 \mapsto v_6$

$u_2 \mapsto v_3$

$u_3 \mapsto v_4$

$u_4 \mapsto v_5$

$u_5 \mapsto v_1$

$u_6 \mapsto v_2$
To see whether a preserved edge write down the adjacency matrices:

\[
\begin{array}{ccccccc}
& u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\
\hline
u_1 & 0 & 1 & 0 & 1 & 0 & 0 \\
u_2 & 1 & 0 & 1 & 0 & 0 & 1 \\
u_3 & 0 & 1 & 0 & 1 & 0 & 0 \\
u_4 & 1 & 0 & 1 & 0 & 1 & 0 \\
u_5 & 0 & 0 & 0 & 1 & 0 & 1 \\
u_6 & 0 & 1 & 0 & 0 & 0 & 1 \\
\end{array}
\]

\[A_6:\]

\[
\begin{array}{ccccccc}
& v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\
\hline
v_6 & 0 & 1 & 0 & 1 & 0 & 0 \\
v_3 & 1 & 0 & 1 & 0 & 0 & 1 \\
v_4 & 0 & 1 & 0 & 1 & 0 & 0 \\
v_5 & 1 & 0 & 1 & 0 & 1 & 0 \\
v_1 & 0 & 0 & 0 & 1 & 0 & 1 \\
v_2 & 0 & 1 & 0 & 0 & 0 & 1 \\
\end{array}
\]

\[A_6:\]
Connected components

**Def.** A walk in a graph $G = (V, E)$ is a non-empty sequence of vertices $W = (v_0, v_1, \ldots, v_k)$ where $k \geq 0$ and $v_i v_{i+1} \in E$ for $i = 0, 1, \ldots, k-1$.

- A walk $W$ connects vertices $v_0$ and $v_k$.
- $W$ is a trail if $v_0, v_1, v_2, \ldots, v_{k-1}, v_k$ are all distinct edges.
- $W$ is a path if vertices $v_0, v_1, \ldots, v_k$ are all distinct.
- $(v_0, v_1, \ldots, v_{k-1}, v_0)$ is a $k$-cycle if vertices $v_0, \ldots, v_{k-1}$ are distinct.
Example.

- Walk \((a, b, c, d, b, e)\) is a trail but not a path.
- \((b, c, d, b)\) is a 3-cycle.
* Vertices \( u \) and \( v \) are connected if there exists a walk with end points \( u \) and \( v \).

* This denoted \( u \sim v \),
  where \( \sim \) is a equivalence relation.

* Graph \( G \) is connected if any two vertices of \( G \) are connected.
A shortest walk between connected vertices is called a geodetic line.

The length of a geodetic line between \( u \) and \( v \) is the distance between \( u \) and \( v \), \( d(u,v) \).

**Lemma.** If \( u \) and \( v \) are connected vertices of a graph with \( n \) vertices, then \( d(u,v) \leq n-1 \).

**Proof.** Using induction on \( n \).

Claim holds if \( n = 2 \).

Assume that claim holds for \( n = k \).

\( G \) has \( k+1 \) vertices:

\[
\text{this part has } k \text{ vertices.}
\]

If \( u \) and \( v \) are connected, then there is an edge \( uv \). The distance \( d(u,v) \leq k-1 \).
Example: Graph $G$:

- $G$ is 3 components

Note: $(\mathcal{L}_G, E_G) = (\mathcal{L}_h, E_h)$
Def: For $G = (V, E)$, each equivalence class $[v]_\sim$ defines a connected subgraph of $G$, $( [v]_\sim, E_v )$ where

$$E_v = \{ uw \in E \mid u \sim v \text{ and } w \sim v \}$$

Subgraphs $( [v]_\sim, E_v )$ are the (connected) components of $G$. 
Multi-graph: allow more than one edge between two vertices

Def: A walk in a multi-graph $G = (V, E)$ is an Eulerian trail if every edge of $G$ appears in the walk exactly once. A closed Eulerian trail is called an Eulerian circuit.
Example:

Eulerian circuit:

(a, b, d, f, e, d, c, e, b, c, a)
Theorem. A connected (multi)graph has an Eulerian circuit iff the degree of every vertex is even.

Proof. Use strong induction on number of edges.

Base case: no edges and graph has to be the trivial graph $K_1$.

Inductively assume that claim holds always when number of edges is $\leq k$.

Consider $G = (V,E)$ that has $k+1$ edges. Graph $G$ must have at least two vertices and this mean that $G$ must have a cycle. C.

Justification: consider an edge $v_0v_1$. Since $v_1$ has even degree, $v_1$ must be incident to some other $v_1v_2$. Repeating this argument we get vertices $v_0, v_1, v_2, v_3, \ldots$ until we come back to a vertex that occurred before.
Remove from $G$ all edges occurring in cycle $C$ and let $G' = (V, E')$ be the resulting graph.
In $G'$ the degree of each vertex is even. $G'$ may not be connected and let $R_1, \ldots, R_s$ be the connected components of $G'$.
By inductive assumption each component $R_i$ has an Eulerian circuit.
Each connected component must have a vertex $v_i$ on cycle $C$. The Eulerian circuit on $R_i$ must visit $v_i$. 
By combining the Eulerian circuits on components $R_1, \ldots, R_s$ with the cycle, we get an Eulerian for $G$.

Note: Edges of $C$ do not belong to any $R_i$. 
Theorem. Let $G$ be a connected graph with exactly two vertices of odd degree, $u$ and $v$. Then $G$ has an Eulerian trail from $u$ to $v$. 
Example:

Eulerian trail: (from $d$ to $b$)

$$(d, f, e, d, c, e, b, c, a, b)$$
Def.: Consider a graph $G = (V, E)$.

A Hamiltonian path visits each vertex of $G$ exactly once.
A Hamiltonian cycle visits each vertex of $G$ exactly once and returns to the start vertex.
Graph $G$ is Hamiltonian if it admits a Hamiltonian cycle.

Note: There is no known characterization of graphs that admit a Hamiltonian cycle/path.

Deciding whether a graph admits a Hamiltonian cycle/path is an algorithmically hard problem (NP-complete).
Example 1.

Hamiltonian cycle $Z$

$(a, b, c, d, e, a)$
Example 2.

Hamiltonian cycle or path?

Hamiltonian path: \((a, b, c, d)\)

Hamiltonian cycle?

A possible Hamiltonian cycle would have to traverse the edge \(ab\) twice. One of vertices \(a\) or \(b\) would have to occur more than once in the cycle. (not including the start & end point)
Summary:

- Eulerian trail/circuit: every edge visited exactly once
- Simple characterization of graphs that admit Eulerian circuit/trail (check parity of degrees of vertices)

- Hamiltonian path/cycle: every vertex visited exactly once
- No known characterization of graphs that admit Hamiltonian path/cycle
- Deciding existence of Hamiltonian path/cycle is algorithmically intractable
- Can use ad hoc techniques to decide whether a graph has a Hamiltonian cycle/path
Example 3.

Is it a Hamiltonian path?

No Hamiltonian path: any walk visiting all vertices a, d, f must traverse one of edges ab, dc, ef more than once.
Example 4.

A Hamiltonian cycle? (or Hamiltonian path?)

Yes: \((a, i, b, c, d, e, f, g, h, a)\)
Example 5.

Hamiltonian cycle or Hamiltonian path?

Hamiltonian path: \((a, b, c, d, e, f, g)\)

No Hamilton cycle: any walk starting from \(d\) has to come back to \(d\) before visiting all other vertices.
Example 6.

Challenge problem: Prove that $G$ does not have a Hamiltonian path.

One of the smallest two-connected graphs with no Hamiltonian path.

* If you feel the assignments are too easy & boring you can work on this question.
Map coloring
US states

How many colors needed? 3
Def. A $k$-coloring of a graph $G = (V, E)$ is a function

$$f : V \rightarrow \{1, \ldots, k\}$$

The coloring is proper if for all $u, v \in V$:

$$\{u, v\} \in E \implies f(u) \neq f(v)$$

Ex.

![Graph diagram](image)
Def: The chromatic number of $G$, $\chi(G)$, is the smallest number of colors needed for a proper coloring of $G$.

Examples: $\chi(K_n) = n$

\[ K_5 \]

$\chi(C_3) = 3$

\[ C_3 \]

$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd}, \quad n \geq 3 \end{cases}$
Proposition. If $\Delta$ is the maximum degree of any vertex in $G$ then

$$\chi(G) \leq \Delta + 1$$

Proposition. If $H$ is a subgraph of $G$, then $\chi(H) \leq \chi(G)$.

Proof. Any proper coloring of $G$ is a proper coloring of $H$. 
Theorem. Graph $G$ is bipartite iff $\chi(G) = 2$.

$G = (V, E)$ is bipartite if $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ and each edge of $E$ has one end-point in $V_1$ and the other in $V_2$.

$K_{3,3}$
Planar graphs

K₅ is planar:

Late 1800's: If G is planar then \( \chi(G) \leq 5 \)

Appel, Haken (1976): For any planar graph G, \( \chi(G) \leq 4 \)