The Combinatorics Cheatsheet

<table>
<thead>
<tr>
<th>Repetition</th>
<th>Order matters</th>
<th>Order doesn't matter</th>
</tr>
</thead>
<tbody>
<tr>
<td>not allowed</td>
<td>$\frac{n!}{(n-k)!}$</td>
<td>$\frac{n!}{k! (n-k)!}$</td>
</tr>
<tr>
<td>allowed</td>
<td>$\frac{k}{n}$</td>
<td>$\frac{(k + n - 1)!}{k! (n-1)!}$</td>
</tr>
</tbody>
</table>

Permutations

Combinations
1. Consider permutations (with no repetition) of the sequence of nine letters ABCDEFGH. Answer the following questions and justify your answers.

(i) How many permutations contain the string FAD?

"FAD" fixed, remaining symbols can go before/after in any order

\{ FAD, B, C, E, G, H^3 \} \to 6 \text{ symbols}, 6! = 720 \text{ permutations}

(ii) How many permutations contain the string BADEF?

"BADEF" fixed, rest of problem similar to (i)

\{ BADEF, C, G, H^3 \} \to 4 \text{ symbols}, 4! = 24 \text{ permutations}

(iii) How many permutations contain both the strings ABC and FGH?

"ABC" and "FGH" fixed; both blocks and remaining symbols can go in any order

\{ ABC, FGH, D, E^3 \} \to 4 \text{ "symbols"}, 4! = 24 \text{ permutations}

(iv) How many permutations contain both the strings CE and DE?

If "CE" and "DE" both appear, then E must be preceded by both C and D simultaneously, which is impossible.

Zero permutations
Short Answer

[10 marks] 2. (a) Using any appropriate proof technique, prove the following statement: for all natural numbers \( m \geq 2 \) and \( n \geq 2 \),

\[
\binom{m+n}{2} - \binom{m}{2} - \binom{n}{2} = mn.
\]

*Hint.* Recall that \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \).

*Use direct proof.* Let \( m, n \in \mathbb{N} \) where \( m, n \geq 2 \).

\[
\binom{m+n}{2} - \binom{m}{2} - \binom{n}{2} = \frac{(m+n)!}{2!(m+n-2)!} - \frac{m!}{2!(m-2)!} - \frac{n!}{2!(n-2)!}
\]

\[
= \frac{(m+n)(m+n-1)}{2} - \frac{m(m-1)}{2} - \frac{n(n-1)}{2}
\]

\[
= \frac{m^2 + 2mn + n^2 - m - n - m(m-1) - n(n-1)}{2}
\]

\[
= \frac{2mn}{2}
\]

\[
= mn
\]

[10 marks] (b) Consider the recurrence relation \( a_n = a_{n-1} + (a_{n-2})^2 + a_{n-3} + 2 \) with initial terms \( a_1 = a_2 = a_3 = 1 \).

i. Calculate the terms \( a_4, a_5, a_6, \) and \( a_7 \).

\[
a_4 = 5
\]

\[
a_5 = 9
\]

\[
a_6 = 37
\]

\[
a_7 = 125
\]

ii. What is the degree of \( a_n \)?

\[ 3 \]

iii. For each of the properties listed below, mark the circle if \( a_n \) satisfies that property.

- ○ \( a_n \) is linear.
- ☒ \( a_n \) is nonlinear.
- ○ \( a_n \) is homogeneous.
- ☒ \( a_n \) is not homogeneous.

Student Number: __________________________
[10 marks] 5. Using any appropriate proof technique, prove the hockey-stick identity: for all \( n, k \in \mathbb{N} \) where \( 0 \leq k < n \),

\[
\sum_{i=0}^{k} \binom{n+i}{i} = \binom{n+k+1}{k+1}.
\]

*Hint.* You may find Pascal's identity useful: \( \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \).

**Use induction.** Let \( P(k) \) be the given statement.

**B.C.** \( P(0) : \sum_{i=0}^{0} \binom{n+i}{i} = \binom{n+0+1}{0} \)

\[
\binom{n}{0} = \binom{n+1}{0} = 1
\]

**I.H.** Assume \( P(k') \) true for some \( k' \in \mathbb{N} \), \( 0 \leq k' < n \).

**I.C.** \( P(k'+1) : \sum_{i=0}^{k'+1} \binom{n+i}{i} = \left( \sum_{i=0}^{k'} \binom{n+i}{i} \right) + \binom{n+k'+1}{k'+1} \) (given)

\[
= \binom{n+k'+1}{k'} + \binom{n+k'+1}{k'+1} \quad \text{(by I.H.)}
\]

\[
= \binom{n+(k'+1)+1}{k'+1} \quad \text{(by P.I.)}
\]

By the PMI, \( P(k) \) is true for all \( k \in \mathbb{N} \), \( 0 \leq k < n \).

[1 mark] **Bonus.** Why is this combinatorial identity called the "hockey-stick identity"?
[10 marks] 4. Recall the Fibonacci sequence $F$, which is the infinite sequence satisfying the following conditions:

- $F_1 = 1$;
- $F_2 = 1$; and
- $F_n = F_{n-1} + F_{n-2}$ for all $n \in \mathbb{N}$ where $n \geq 3$.

Using any appropriate proof technique, prove that for all $n \in \mathbb{N}$ where $n \geq 1$,

$$F_n + 2F_{n+1} = F_{n+4} - F_{n+2}.$$

**Use strong induction. Let $P(n)$ be the given statement.**

**B,C.1. $P(1):$**

$$F_1 + 2F_2 = F_5 - F_3$$

$$1 + 2(2) = 5 - 2$$

$$3 = 3$$

**B,C.2. $P(2):$**

$$F_2 + 2F_3 = F_6 - F_4$$

$$1 + 2(3) = 8 - 3$$

$$5 = 5$$

**I.H. Assume $P(k')$ true for all $k' \in \mathbb{N}$, $3 \leq k' \leq k$.**

**I.C. $P(k+1):$** By the inductive hypothesis, we know

$$F_k + 2F_{k+1} = F_{k+4} - F_{k+2} \text{ and}$$

$$F_{k-1} + 2F_k = F_{k+3} - F_{k+1}.$$ 

Sum these expressions and group like terms to get

$$(F_k + F_{k-2}) + (2F_{k+1} + 2F_k) = (F_{k+4} + F_{k+3}) - (F_{k+2} + F_{k+1})$$

$$F_{k+2} + 2F_{k+2} = F_{k+5} - F_{k+3}.$$ 

By the strong PMI, $P(n)$ is true for all $n \in \mathbb{N}$, $n \geq 1$. 

\[\blacksquare\]
Example. A fair coin is tossed six times. 

$X$ is a random variable that gives the number of heads.

Calculate $E(X)$.

\[
\begin{array}{c|ccccccc}
\text{value of } X & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
\text{probability} & \frac{1}{64} & \frac{6}{64} & \frac{15}{64} & \frac{20}{64} & \frac{15}{64} & \frac{6}{64} & \frac{1}{64} \\
\end{array}
\]

\[
\binom{6}{2} = 15
\]

\[
E(X) = \sum_{s \in S} X(s) P(s) = \ldots = 3
\]

\underline{SIMPLER SOLUTION:}

Define: $X_i$ gets the number of heads of $i^{th}$ coin toss. ($X_i$ has value 0 or 1)

\[
E(X_i) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0 = \frac{1}{2}
\]

\[
X = X_1 + X_2 + \ldots + X_6
\]

\[
E(X) = \sum_{i=1}^{6} E(X_i) = 3
\]
Example (cont.) \( X \) is the number of times heads occurs when a fair coin is tossed six times.

Calculate the variance of \( X \).

\[
V(X) = E(X^2) - E(X)^2
\]

\[
E(X^2) = 0^2 \cdot \frac{1}{64} + 1^2 \cdot \frac{6}{64} + 2^2 \cdot \frac{15}{64} + 3^2 \cdot \frac{20}{64} \\
+ 4^2 \cdot \frac{15}{64} + 5^2 \cdot \frac{6}{64} + 6^2 \cdot \frac{1}{64} = \ldots = 10.5
\]

\[
E(X)^2 = 9
\]

\[
V(X) = 1.5
\]

As in previous example

\( X = X_1 + X_2 + \ldots + X_6 \)

\[
V(X_i) = E(X_i^2) - E(X_i)^2
\]

\[
= \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \quad E(X_i^2) = \frac{1}{2} \cdot 1^2 + \frac{1}{2} \cdot 0^2
\]

Because \( X_i \)'s are independent:

\[
V(X) = \sum_{i=1}^{6} V(X_i) = 1.5
\]