Permutation: an ordered re-arrangement of elements of a finite set

Example. \( A = \{a, b, c, d\} \)

Permutations of \( A \):

\[
\begin{align*}
\text{abcd} & | \text{bacd} & | \text{cabd} & | \text{dabc} \\
\text{abdc} & | \text{badc} & | \text{cadb} & | \text{dacb} \\
\text{acbd} & | \text{bcad} & | \text{cbad} & | \text{dbac} \\
\text{acdb} & | \text{bcda} & | \text{cbda} & | \text{dbca} \\
\text{adbc} & | \text{bdac} & | \text{cdab} & | \text{dcab} \\
\text{adcb} & | \text{bdca} & | \text{cdba} & | \text{dcba}
\end{align*}
\]
A 4-element set has $4! = 24$ permutations.

An $n$-element set has $n!$ permutations.

Formally, a permutation of a finite set $A$ is a bijection $A \rightarrow A$. 
Example: Two line notation

\[ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \]

\[ \sigma(1) = 3 \]
\[ \sigma(2) = 1 \]
\[ \sigma(3) = 4 \]
\[ \sigma(4) = 2 \]

\( \sigma \) is a cycle:

\[ 1 \xrightarrow{\sigma} 3 \xrightarrow{\sigma} 4 \xrightarrow{\sigma} 2 \xrightarrow{\sigma} 1 \]

\( \sigma \) in cycle notation:

\( (1, 3, 4, 2) \)
Example: 
\[ \pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 7 & 5 & 6 & 3 & 8 & 1 & 4 & 9 \end{pmatrix} \]

1 → 2 → 7 → 1
3 → 5 → 3
4 → 6 → 8 → 4
9 → 9

Any permutation can be written as a product of cycles:

\[ \pi = (1, 2, 7)(3, 5)(4, 6, 8)(9) \]

"cycle notation"

Inverse of \( \pi \):

\[ \pi^{-1} = (7, 2, 1)(5, 3)(8, 6, 4)(9) \]
Generalizations of permutation

An ordered arrangement of \( k \) elements from an \( n \) elements set \((k \leq n)\) is a \( k \)-permutation.

The number of \( k \) permutations of an \( n \)-element set is \( P(n, k) \).

Claim. \( P(n, k) = \frac{n!}{(n-k)!} \)

Proof. Choices for 1st element; \( n \)
- 1st = 2nd = 3rd = \( \ldots \) = \( n-1 \)
- 1st = 2nd = 3rd = \( \ldots \) = \( n-2 \)
  
  \( \vdots \)
- 1st = \( k^{th} \) = \( \ldots \) = \( n-k+1 \)

\[ P(n, k) = n \cdot (n-1) \cdot \ldots \cdot (n-k+1) = \frac{n!}{(n-k)!} \]
Example. \( A = \{1, 2, 3\} \)

List all 2-permutations of A.

\[
\begin{align*}
1, 2 \\
2, 1 \\
1, 3 \\
3, 1 \\
2, 3 \\
3, 2
\end{align*}
\]

\[
P(3, 2) = \frac{3!}{1!} = 6
\]

Note: \( P(n, k) \) is the number of one-to-one functions from a \( k \)-element set to an \( n \)-element set, \( (k \leq n) \).
Example: Olympic marathon has 100 competitors. In how many ways can we select the three medal winners? (gold, silver, bronze)

Number is $P(100, 3) = 100 \cdot 99 \cdot 98$
Example: How many permutations of the letters A, B, C, D, E, F, G, H contain the string ABC?

We view "ABC" as one block. Permute objects: "ABC", "D", "E", "F", "G", "H"

Number of permutations:

6! = 720
Def: $k$-permutation on an $n$-element set $\mathcal{L}$ with repetition allowed:
the number of choices $n^k$

Note: we allow $k > n$

Example: The number of strings of length $k$ over the English alphabet: $26^k$
Combinations

A k-combination of a set is an unordered selection of k elements from the set.

Example. 3-combinations of the set \{1, 2, 3, 4\}

\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}
Number of $k$-combinations of an $n$-element set

$C(n, k)$ also denoted $\binom{n}{k}$ (binomial coefficient)

Recall from last week:

$P(n, k) = \text{number of } k\text{-permutations of an } n\text{-element set}$

$$P(n,k) = \frac{n!}{(n-k)!}$$

$$C(n, k) = \frac{P(n,k)}{P(k,k)} = \frac{n!}{k! \cdot (n-k)!}$$
Example: How many poker hands of five cards can be dealt from a deck of 52 cards?

Answer: \( \binom{52}{5} = \frac{52!}{5! \cdot 47!} \)

\[ \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5!} = 2,598,960 \]
Example. How many strings are reorderings of the letters in $\text{SUCCESS}$?

$S_1\, U\, C_1\, C_2\, E\, S_2\, S_3$

$S_2\, E\, C_2\, C_1\, U\, S_3\, S_3$ \{ different permutations that are not distinguishable! \}

$S_1\, E\, C_1\, C_2\, U\, S_2\, S_3$ \{ different permutations that are not distinguishable! \}

Count the number of distinguishable permutations!
3 occurrences of S
2 " = c
1 " = u, e

^ # of ways to select positions of S
(7) \cdot (4) \cdot (2) \cdot (1) =
(3) \cdot (2) \cdot (1) \cdot (1)

^ # of ways to position C's in "empty places"

\frac{7!}{3! \cdot 4!} \cdot \frac{4!}{2! \cdot 2!} \cdot \frac{2!}{1! \cdot 1!} \cdot \frac{1!}{1!}

= \frac{7!}{3! \cdot 2!} = 420
Permutations with indistinguishable objects

Number of different permutations of \( n \) objects where

- \( n_1 \) objects of type 1
- \( n_2 \) objects of type 2
- \( \ldots \)
- \( n_k \) objects of type \( k \)

\[ \sum_{i=1}^{k} n_i = n \]

\[
\begin{align*}
\left( \begin{array}{c} n \\ n_1 \end{array} \right) \cdot \left( \begin{array}{c} n-n_1 \\ n_2 \end{array} \right) \cdot \left( \begin{array}{c} n-n_1-n_2 \\ n_3 \end{array} \right) \ldots \left( \begin{array}{c} n-n_1-\ldots-n_{k-1} \\ n_k \end{array} \right) \\
\end{align*}
\]

\[
= \frac{n! \cdot (n-n_1)! \cdot (n-n_1-n_2)! \ldots (n-n_1-\ldots-n_{k-1})!}{n_1! \cdot (n-n_1)! \cdot n_2! \cdot (n-n_1-n_2)! \ldots n_k!}
\]

\[
= \frac{n!}{n_1! \cdot n_2! \ldots n_k!}
\]
Combinations with repetition

A $k$-combination of a set of $n$ elements when repetition is allowed:

- select $k$ elements, possibly same element multiple times
- order does not matter
Example. (Stars and bars method)

Tim Horton's at Bioscience has 4 different varieties of cookies.

In how many ways can 6 cookies be chosen?

- only the type of the cookie matters
- individual cookies do not matter
- the order does not matter
4 - 1 = 3 bars
6 stars

***
\[\text{type 1 cookies}\]

*
\[\text{type 2 cookies}\]

*
\[\text{type 3 cookies}\]

***
\[\text{type 4 cookies}\]

String of length \((4 - 1) + 6\) represents an selection of 6 cookies
\[
\left(6 + 4 - 1\right)! \quad \frac{9 \cdot 8 \cdot 7}{3 \cdot 2} = 84
\]

\[
\frac{6! \cdot 3!}{3! \cdot 2!}
\]

↑

"names" of the bar-symbols do not matter

all cookies are indistinguishable
The number of k-combinations of a set of n elements with repetition allowed:

\[ \frac{(k+n-1)!}{k! (n-1)!} \]
**Summary:** combinations and permutations of an n-element set

**FORMULA**

\[ P(n, k) = \frac{n!}{(n-k)!} \]

\[ C(n, k) = \frac{n!}{k!(n-k)!} \]

- **k-permutations**
  - with repetition
    \[ n^k \]

- **k-combinations**
  - with repetition
    \[ \frac{(k + n - 1)!}{k! (n-1)!} \]
Examples:

(i) How many bit strings of length 10 contain exactly four 1's?

\[ \binom{10}{4} \]

Selecting the 4 positions of 1's completely determine the bit string.

(ii) How many strings of length 10 are over the English alphabet?

Alphabet size = 26

\[ 26^{10} \quad (10\text{-permuatation with repetition}) \]
(iii) A 1500m run has 20 competitors.
In how many ways can we select the gold, silver, bronze winners?

\[ P(20,3) = 20 \cdot 19 \cdot 18 \]

(iv) Tim Hortons has 5 varieties of donuts.
In how many ways can we select a dozen donuts?

\[ \frac{(12 + 5 - 1)!}{12! \cdot (5-1)!} \]
Multisets "bags"

\[ \{ 1, 2, 2 \} : \text{ one occurrence of 1, two occurrences of 2 } \]

\[ \{ 1, 2, 2 \} \neq \{ 1, 2 \} \]

Multiset is unordered:

\[ \{ 1, 2, 2 \} = \{ 2, 1, 2 \} \]
Note: a k-combination of a set of n-elements with repetition allowed can be viewed as a multiset of cardinality k whose elements belong to \{1, \ldots, n\}.

Example: underlying set \{1, 2, 3, 4, 5, 6\}

Multiset:
\{1, 2, 2, 2, 3, 3, 5, 5, 5, 6\}

Stars and bars:

\[
\begin{array}{|c|c|c|c|}
\hline
\ast & \ast\ast\ast & \ast\ast & \ast\ast\ast\ast\ast \\
\hline
\end{array}
\]
Textbook notation:
\[
\binom{n}{k} = \text{number of multisets of cardinality } k \text{ when underlying elements belong to } \{1, \ldots, n\}
\]

Th. 18.8
\[
\binom{n}{k} \downarrow \binom{n+k-1}{k}
\]

\[
= \frac{(k+n-1)!}{k!(n-1)!} = \text{number of } k\text{-combinations of an } n\text{-element set with repetition allowed}
\]
Binomial theorem

\[(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j\]

\[= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \ldots + \binom{n}{n} y^n\]
Lemma. For $n \geq 0$:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof:

$$\begin{align*}
(1+1)^n &= \sum_{j=0}^{n} \binom{n}{j} 1^{n-j} y^j \\
&= \sum_{j=0}^{n} \binom{n}{j}
\end{align*}$$

Explanation:

$\binom{n}{k}$ = number of subsets of size $k$

$2^n$ = number of all subsets