Example: Sum of the values that appear on two independent dice.

\( X_1 \): the number appearing on first die

\( X_2 \): second die

\[ E(X_1) = \frac{1}{6} (1 + 2 + 3 + 4 + 5 + 6) = \frac{21}{6} = \frac{7}{2} \]

\[ E(X_2) = \frac{7}{2} \text{ (similarly)} \]

\[ E(X_1 + X_2) = E(X_1) + E(X_2) = 7 \]
Do we have a similar product rule:

\[ E(X \cdot Y) = E(X) \cdot E(Y) \, ? \]

Answer: no, in general

Counter-example:

Coin is flipped two times

- \( X \): number of "heads"
- \( Y \): number of "tails"

\[
E(X) = \frac{1}{4} \left[ X(HH) + X(HT) + X(TH) + X(TT) \right]
\]

\[ = \frac{1}{2} \]

\[
E(Y) = \frac{1}{4} \left[ Y(HH) + Y(HT) + Y(TH) + Y(TT) \right]
\]

\[ = \frac{1}{2} \]

\[
E(XY) = \frac{1}{4} \left[ X(HH)Y(HH) + X(HT)Y(HT)
+ X(TH)Y(TH) + X(TT)Y(TT) \right]
\]

\[ = \frac{1}{2} \]

\[ = \frac{1}{2} \]
Independent random variables

- \( X, Y \) random variables on sample space \( S \)

Definition. \( X \) and \( Y \) are independent if for all \( s \in S \) and all values \( v_1, v_2 \):

\[
P(X(s) = v_1 \text{ and } Y(s) = v_2) = P(X(s) = v_1) \cdot P(Y(s) = v_2)
\]
Theorem. If $X$ and $Y$ are independent random variables, then $E(XY) = E(X) \cdot E(Y)$

Proof in the textbook.
Example: Pair of fair dice.

\[ X_1(s): \text{value of first die} \]
\[ X_2(s): \text{value of second die} \]

\( X_1 \) and \( X_2 \) are independent.

\[ P(X_1(s) = a \text{ and } X_2(s) = b) = \frac{1}{36} \]
\[ = P(X_1(s) = a) \cdot P(X_2(s) = b) \]

Expected value of product:

\[ E(X_1 \cdot X_2) = E(X_1)E(X_2) = \left(\frac{7}{2}\right)^2 = \frac{49}{4} \]
Example. Markov's inequality (exercise in text)

Let $X$ be a random variable that takes only nonnegative values.

Show that

$$P[X \geq a] \leq \frac{E(X)}{a}, \quad (a > 0)$$

Proof. Define a 0-1 valued random variable $Y$:

$$Y(s) = \begin{cases} 1 & \text{if } X(s) \geq a \\ 0 & \text{if } X(s) < a \end{cases}$$

Note: for all $s \in S$; $X(s) \geq a \cdot Y(s)$

Estimate

$$E(X) \geq E(a \cdot Y) = a \cdot E(Y)$$

$\uparrow$

linearity

$$= a \cdot \sum_{s \in S} P(s)Y(s) = a \cdot P[X \geq a]$$

$$\underbrace{P[X \geq a]}_{P[X \geq a]}$$
Variance

- The expected value of a random variable gives its weighted average, but not how widely the values are distributed.
- The variance of a random variable characterizes how widely the values are distributed.
Example: Sum of values of two dice

2: 2 2  (1/36)
3: 3 3 3 3 3 (2/36)
4: 4 4 4 4 4 4 (3/36)
5: 5 5 5 5 5 5 5 5 5 (4/36)
6: 6 6 6 6 6 6 6 6 6 6 6 (5/36)
7: 7 7 7 7 7 7 7 7 7 7 7 7 7 7 7 7 7 7 7 7 (6/36)
8: 8 8 8 8 8 8 8 8 8 8 8 8 8 8 (5/36)
9: 9 9 9 9 9 9 9 9 9 9 9 (4/36)
10: 10 10 10 10 10 10 10 (3/36)
11: 11 11 11 11 11 11 (2/36)
12: 12 12 (1/36)
Example: Sum of values of two dice

2:  (1/36)
3:  (2/36)
4:  (3/36)
5:  (4/36)
6:  (5/36)
7:  (6/36)
8:  (5/36)
9:  (4/36)
10: (3/36)
11: (2/36)
12: (1/36)
How to define variance?

- model the distance of an outcome from expectation:
  for each \( s \in S \): how far is \( X(s) \) from \( E(X) \)
- each outcome is weighted by its probability

Incorrect attempt:

\[
\sum_{s \in S} P(s) \left( X(s) - E(X) \right)
\]

\[
= \sum_{s \in S} P(s) X(s) - \left( \sum_{s \in S} P(s) \right) E(X) = 0
\]

\[= E(X) = 1 \]
Definition. Let $X$ be a random variable in sample space $S$.

The variance of $X$ is

$$V(X) = \sum_{s \in S} (X(s) - E(X))^2 \cdot P(s)$$

The standard deviation of $X$:

$$\sigma(X) = \sqrt{V(X)}$$
Theorem. For a random variable $X$ on sample space $S$:

$$V(X) = E(X^2) - E(X)^2$$

Proof. 

$$V(X) = \sum_{s \in S} (X(s) - E(X))^2 \cdot P(s)$$

$$= \sum_{s \in S} X(s)^2 \cdot P(s) - 2 E(X) \sum_{s \in S} X(s) \cdot P(s)$$

$$+ E(X)^2 \sum_{s \in S} P(s)$$

$$= \sum_{s \in S} X(s)^2 \cdot P(s) - 2 E(X)^2 + E(X)^2 = E(X^2) - E(X)^2$$
Example. The variance of the value of a die. 

Random variable $X$: the number that comes up

Earlier $E(X) = \frac{7}{2}$

$E(X^2) = \frac{1}{6} \left[ 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 \right] = \frac{91}{6}$

$V(X) = \frac{91}{6} - \left( \frac{7}{2} \right)^2 = \frac{182 - 3,49}{12} = \frac{35}{12}$
Monty Hall problem

- Game show contestant is asked to choose one of three doors 1, 2, or 3.
- Behind one of the doors is a car, and behind the two others a goat.
- After contestant chooses a door, the host opens one of the remaining doors and shows behind it a goat.
- The contestant is offered the opportunity to switch to the other remaining closed door.
- Should they accept?

Options:

- Yes
- No
- Doesn't matter

Answer: Yes

Marked as correct.
Due to symmetry consider case where contestant chooses door 1
"C1:02" : car behind door 1, host opens door 2
Sample space:
C1:02, C1:03
C2:03, C3:02
P(C1:02) = \frac{1}{6}, P(C1:03) = \frac{1}{6}
P(C2:03) = \frac{1}{3}, P(C3:02) = \frac{1}{3}
Consider the case where host opens door 2 (other case is symmetric)

Probabilities we should compute:

"car behind door 1 on condition host opens door 2"

\[ P(A \mid C) \]

and

"car behind door 3 on condition host opens door 2"

\[ P(B \mid C) \]

Events:

- car behind door 1: \( A = \{ C1:02, C1:03 \} \)
- car behind door 3: \( B = \{ C3:02 \} \)
- host opens door 2: \( C = \{ C1:02, C3:02 \} \)

\[ P(A) = P(B) = \frac{1}{3} \]
\[ P(A \mid C) = \frac{P(A \cap C)}{P(C)} \]

\[ P(A \cap C) = P(C1 : 02) = \frac{1}{6} \]

\[ P(C) = P(C1 : 02) + P(C3 : 02) = \frac{1}{6} + \frac{1}{3} = \frac{1}{2} \]

\[ P(A \mid C) = \frac{1}{6} \div \frac{1}{2} = \frac{1}{3} \]

---

**P(B \mid C):**

\[ P(B \cap C) = P(C3 : 02) = \frac{1}{3} \]

\[ P(C) = \frac{1}{2} \]

\[ P(B \mid C) = \frac{1}{3} \div \frac{1}{2} = \frac{2}{3} > P(A \mid C) \]
Partially ordered sets

A relation \( \leq \) on a set \( A \) is a partial order if it is:

- reflexive, \( (\forall a \in A \quad a \leq a) \)
- antisymmetric, and, \( (a \leq b \text{ and } b \leq a \text{ implies } a = b) \)
- transitive, \( (a \leq b \text{ and } b \leq c \text{ implies } a \leq c) \)

\( (A, \leq) \) is a partially ordered set (poset)

- a and b are comparable if \( a \leq b \) or \( b \leq a \)
- otherwise a and b are incomparable, a \( \nmid \) b

A special case: \( \leq \) is a total order if any two elements are comparable
Examples.

(i) $S$ is a set of tasks in a project. Define $t_1 \leq t_2$ if $t_1$ must be completed before $t_2$ begins.

(ii) $A$ is a set

\[
(\mathbb{Z}^A, \leq)\]

Ex. $A = \{1, 2\}$

\{1\}, \{2\} are incomparable

(iii) \((\mathbb{Z}, \leq)\) "less than equal to"

total order

(iv) divisibility relation in \(\mathbb{N}\)

\(a \mid b\) if \((\exists c \in \mathbb{N}) a \cdot c = b\)

\((\mathbb{N}, \mid)\) is a poset

2 and 3 are incomparable
Maximal and minimal elements

\((A, \leq)\) is a poset

- \(c \in A\) is maximal if \(c \preceq A\) and \((\forall a \in A)\ c \leq a\) implies \(c = a\)
- \(c \in A\) is minimal if \((\forall a \in A)\ a \leq c\) implies \(c = a\)
Example.

\[
\left( \{2, 4, 5, 10, 12, 20, 25\}, \mid \right)
\]

Hasse diagram

12 → 20
4 → 10 → 25
2 → 5

- maximal elements: 12, 20, 25
- minimal elements: 2, 5
- no greatest or least element

edge upward from \( x \) to \( y \): \( x \mid y \)

divisibility relation \( \sim \)
\((A, \leq)\) is poset if
- \(c \in A\) is the greatest if \((\forall a \in A)\) \(a \leq c\)
- \(c \in A\) is the least element if \((\forall c \in A)\) \(c \leq a\)

**Example:** \((2^A, \leq)\) \(A = \{a, b, c\}\)

Hasse diagram:

```
{a, b, c}  \\
|   |   |
{a, b} \{a, c\} \{b, c\}
|   |   |
{a} \{b\} \{c\}
|   |
\emptyset
```

greatest element: \(\{a, b, c\}\)
least element: \(\emptyset\)
\((A, \leq)\) poset

- \(c\) is an upper bound for \(A\) if \((\forall a \in A) a \leq c\)
- \(c\) is a lower for \(A\) if \((\forall a \in A) c \leq a\)

\(d\) is the least upper bound for \(A\) if it is the smallest element in the set of upper bounds for \(A\) (supremum)

\(d\) is the greatest lower bound for \(A\) if it is the greatest element in the set of lower bounds for \(A\) (infimum)

Note: A supremum (or infimum) is unique if it exists.
Example. poset \((\mathbb{N} - \{0\}, \leq)\)

Greatest lower bound and least upper bound of \(A = \{3, 9, 12\}\)

- Lower bounds for \(A\): 1, 3
  greatest lower bound: 3

- Upper bounds for \(A\):
  if \(x\) is an upper bound then \(3|x, 9|x, 12|x\)
  least upper bound for \(A\): 36 = \(\text{lcm}(3, 9, 12)\)