CISC 203

Guest Lecture

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Midterms:

Thurs. 11:30 office hrs.

Fri. end of class

Pick up from Prof. Salomaa's office hours
Embeddings (on a 2D plane):

Def. An embedding is a graph where

1. Each vertex is assigned a unique pt. on the surface
2. Each edge is assigned a unique curve on the surface
3. No vertex other than the two incident vertices of an edge lie on that edge.
4. The endpts. of an edge are exactly the two vertices to which it is incident.
Examples:

A graph embedding may have edges that intersect, but do not overlap.

Planar graphs:

Def. A planar graph is a graph embedding that is crossing-free (no intersections).
Examples (planar or not?):

(Exercise!)

Example (utility puzzle):
Faces of graphs:

Every graph has
- a set of vertices, \( V \)
- a set of edges, \( E \)
- a set of faces, \( F \)

Examples:

1) \[ 
\begin{array}{c}
\text{\( f_1 \)} \\
\text{\( f_2 \)} \\
\text{\( f_3 \)} \\
\text{\( f_4 \)} \\
\end{array} 
\]

2) \( f_1 \) (Exterior face)
Theorem (Euler's Formula):

Given a connected planar simple graph $G = (V,E)$ with $v$ vertices, $e$ edges, and $F$ faces,

$$V - e + F = 2.$$  

↑ characteristic

Proof. By induction. (on $|E| = e$)

BC. $|E| = 0$

$v = 1$

$e = 0$

$F = 1$

$$1 - 0 + 1 = 2$$

IH. Assume true for some $|E| = e' \in \mathbb{N}$.

IC. Show that it is true for $|E| = e' + 1$.

(Add one edge to $E$.)
Case 1. New edge incident to one existing vertex $(|E| = e' + 1)$.

We must add a new vertex $(|V| = v + 1)$.

No change to faces $(|F| = F)$.

$$(v + 1) - (e' + 1) + F \leq 2$$

$$v + 1 - e' - 1 + F \leq 2$$

$$v - e' + F = 2$$

Case 2. New edge incident to two existing vertices $(|E| = e' + 1)$.

We don't need a new vertex $(|V| = v)$.

# of faces increases by one $(|F| = F + 1)$.

$$v - (e' + 1) + (F + 1) \leq 2$$

$$v - e' - 1 + F + 1 \leq 2$$

$$v - e' + F = 2$$
Theorem. $K_5$ is nonplanar.

Proof. Assume by contradiction that $K_5$ is planar.

If $K_5$ is planar, then it satisfies Euler's formula.

$|V|_{K_5} = 5$, $|E|_{K_5} = 10$

$5 - 10 + F = 2$

$F = 2 - 5 + 10 = 7$

Let $b$ denote the number of "boundary edges" surrounding each face of $K_5$. 
**Lemma.** Any face in a graph will be surrounded by at least three boundary edges.

This implies that, in $K_5$, we have $b \geq 3F$.

**Lemma.** Each edge in a graph acts as a boundary edge for exactly two faces (i.e., $b = 2e$).

This implies that, in $K_5$, we have $2e \geq 3F$.

In $K_5$, we have $|E| = 10$ and $|F| = 7$. This asserts $2(10) \geq 3(7)$.

Since this is a contradiction, $K_5$ is non-planar.
Theorem. \( K_{3,3} \) is nonplanar.

Proof. Assume by contradiction that \( K_{3,3} \) is planar.

If \( K_{3,3} \) is planar, then it satisfies Euler's formula.

\[ |V|_{K_{3,3}} = 6, \quad |E|_{K_{3,3}} = 9 \]

\[ 6 - 9 + F = 2 \]

\[ F = 2 - 6 + 9 = 5 \]

Lemma. \( K_{3,3} \) does not contain the cycle graph \( C_3 \) as a subgraph.

Let \( b \) again denote the number of boundary edges surrounding each face of \( K_{3,3} \).

By the lemma, each face of \( K_{3,3} \) is surrounded by at least four boundary edges.

\[ b \geq 4f. \]
From before (second lemma of previous proof), we know that $b = 2e$.

$$2e \geq 4f.$$ 

In $K_{3,3}$, we have

$$z(9) \geq y(5).$$

Since this is a contradiction, $K_{3,3}$ is nonplanar.
Theorem (Kuratowski):

A graph $G$ is non-planar if and only if the graph contains a subdivision of $K_5$ or $K_{3,3}$ as a subgraph.

Proof. Omitted. (Way too long!)