

Independent Events and Random Variables

Independent Events

Our intuitive understanding of the word “independent” is “not controlled or influenced”, and this is exactly the sense in which we use the word when discussing probability. We say that event A is **independent** of event B if the probability that A has occurred is not affected by whether B has occurred or not.

For example, suppose our experiment consists of tossing a red die and a blue die (both 6-sided, both fair) Each outcome consists of an ordered pair (r,b) where r is the value of the red die and b is the value of the blue die. We intuitively accept that the two dice are independent of each other so the probability of an event such as “the red die is showing 3” should not be affected by whether or not an event such as “the blue die is showing 4” has occurred.

Consider these two events: A: { the sum of the two dice = 7 } and B: { the blue die is showing either 2 or 3 }

The possible outcomes can be seen in this table, in which the first number in each pair represents the red die, and the second represents the blue die:

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Here the outcomes that constitute A are coloured green and yellow, and the outcomes that constitute B are coloured blue and yellow.

We can see that $P(A) = \frac{6}{36} = \frac{1}{6}$

Now, what is the probability that A has occurred, if we know that B has occurred? B contains 12 outcomes, and 2 of those are in A ... so the probability that A has occurred, given that B has occurred, is $\frac{2}{12} = \frac{1}{6}$ which is exactly $P(A)$!!! Knowing that B has occurred does not increase or decrease the probability that A has occurred. This becomes our mathematical way of expressing the idea that A is independent of B.

In notation, this would mean that $P(A | B) = P(A)$ which we can process as follows

$$P(A|B) = P(A)$$

$$\frac{P(A \cap B)}{P(B)} = P(A)$$

$$P(A \cap B) = P(A) * P(B)$$

Note that we can go one step further:

$$P(A \cap B) = P(A) * P(B)$$

$$\frac{P(A \cap B)}{P(A)} = P(B)$$

which is just the notational way of saying that B is independent of A. In other words, **A is independent of B if and only if B is independent of A.**

Thus we say **A and B are independent events if $P(A \cap B) = P(A) * P(B)$**

Equivalently, if we believe that A and B are independent events, then we expect that $P(A \cap B) = P(A) * P(B)$

Example: Suppose we know A and B are independent events, and we also know $P(A \cap B) = 0.7$, $P(A) = 0.9$. Can we determine $P(B)$?

Yes, we can ... we know $P(A \cap B) = P(A) * P(B)$, so $P(B) = \frac{7}{9}$

Random Variables

Let (S,P) be a sample space. A **random variable** is a function X from S to V where V is some set.

Usually V is a set of numbers, and X is a function that measures some attribute of the outcome.

Example: Let $S = \{\text{result of tossing a red die and a blue die}\} = \{(1,1), \dots, (6,6)\}$

Let $X(s) = \text{the sum of the two values in } s$

So $X((3,2)) = 5$, $X((4,6)) = 10$, etc.

Another Example: Let $S = \{\text{result of tossing a coin 5 times}\} = \{\text{HHHHH, HHHHT, \dots, TTTTT}\}$

Let $X(s) = \text{the number of Heads in } s$

So $X(\text{HHHHH}) = 5$, $X(\text{THHTT}) = 2$, etc.

Yet Another Example: Let $S = \{10,11,12,13,14,15\}$ and let $P(i) = 1/6 \ \forall i \in S$

Let $X(s) = s \% 4$ (where % means “mod”)

s	X(s)
10	2
11	3
12	0
13	1
14	2
15	3

Let's pursue this example a bit further. We can define events based on X . For example, “ $X = 2$ ” is an event, as is “ $X \in \{1,2\}$ ”, “ $X \geq 2$ ”, etc. And since we have events, it makes sense to determine the probability of these events. So what is $P(X = 2)$?

Looking at the table above, we can see that $X(s) = 2$ occurs when $s \in \{10,14\}$, and since 10 and 14 have combined probabilities $= \frac{1}{3}$, we can see that $P(X = 2) = \frac{1}{3}$