

## Random Variables continued

Now let's look at the coin flipping example described previously:

Let  $S = \{\text{result of tossing a coin 5 times}\} = \{\text{HHHHH, HHHHT, ..., TTTTT}\}$

Let  $X(s) = \text{the number of Heads in } s$

So  $X(\text{HHHHH}) = 5$ ,  $X(\text{THHTT}) = 2$ , etc.

Now we can ask questions such as "What is the probability that  $X = 2$ ?" or equivalently, "What is  $P(X = 2)$ ?"

At this point, many people (who should know better) make a crucial mistake: they assume without any justification that the coin we are flipping is balanced (i.e. H and T have equal probability of coming up on each toss). We can't make that assumption, so we deal with the more general case. Suppose the coin has probability  $p$  of coming up H, and therefore probability  $(1-p)$  of coming up T.

What is  $P(X = 2)$  with  $X(s)$  defined as the number of Heads in  $s$ ?

The coin tosses are independent (the coin has no memory), so the probability of tossing the outcome HHTTT (2 heads and then 3 tails) is

$$p \cdot p \cdot (1 - p) \cdot (1 - p) \cdot (1 - p)$$

This is one of the outcomes that gives  $X = 2$

But so is TTHTH, which has probability  $(1 - p) \cdot (1 - p) \cdot p \cdot (1 - p) \cdot p$

and in fact we can see that every outcome containing exactly 2 H's will have probability  $p^2 \cdot (1 - p)^3$ , and  $P(X=2)$  will just be the sum of all of these. How many are there?

The two heads can occur in any of the five positions, so a simple counting argument tells us there are  $\binom{5}{2}$  different outcomes with exactly 2 H's. The final answer is:

$$P(X = 2) = \binom{5}{2} \cdot p^2 \cdot (1 - p)^3$$

Similarly we can compute  $P(X = 4) = \binom{5}{4} \cdot p^4 \cdot (1 - p)$

Exercise: Suppose we are tossing a coin with  $P(H) = 1/3$ ,  $P(T) = 2/3$ . If we toss the coin four times and  $X(s)$  is the number of Heads we see in outcome  $s$ , what is  $P(X=2)$ ?

### Independent Random Variables

Recall that we say the two events  $A$  and  $B$  on sample space  $(S,P)$  are independent if  $P(A \cap B) = P(A) \cdot P(B)$ . Remember that this means that knowing  $A$  has occurred does not affect the probability that  $B$  has occurred (ie  $P(B | A) = P(B)$ ), and vice versa.

We say random variables  $X$  and  $Y$  are *independent* if

$$\forall a \text{ and } b, \quad P(X(s) = a \text{ and } Y(s) = b) = P(X = a) \cdot P(Y = b)$$

This deserves some careful explanation. We are referring to an outcome  $s$  of the experiment, and asking for the probability that  $X(s) = a$  and  $Y(s) = b$ . Here  $a$  and  $b$  are possible values that  $X$  and  $Y$  can give. We can only say  $X$  and  $Y$  are independent if the equation shown above holds for *all possible values* for  $a$  and  $b$ .

This will be clarified by some simple examples.

Example 1: We return to one of our previous examples:  $S = \{10,11,12,13,14,15\}$  and

$$P(s) = \frac{1}{6} \quad \forall s \in S$$

Let  $X$  be a random variable defined on  $(S,P)$  by  $X(s) = s \% 4$

Let  $Y$  be a random variable defined on  $(S,P)$  by  $Y(s) = s \% 5$

Here is the situation:

s	X(s)	Y(s)
10	2	0
11	3	1
12	0	2
13	1	3
14	2	4
15	3	0

Are X and Y independent?

If they are then the equation  $P(X(s) = a \text{ and } Y(s) = b) = P(X = a) \cdot P(Y = b)$  must hold for all values of a and b

Consider  $a = 0, b = 0$ .  $P(X = 0) = \frac{1}{6}$  and  $P(Y = 0) = \frac{2}{6} = \frac{1}{3}$

There is no s such that  $X(s) = 0$  **and**  $Y(s) = 0$ , so  $P(X(s) = 0 \text{ and } Y(s) = 0) = 0$

But  $P(X=0) * P(Y=0) = \frac{1}{18}$  So the equation does not hold for  $a = 0, b = 0$  .... so X and Y are **not independent**.

Exercise : Let S be the set of possible outcomes of tossing a coin 5 times. Let X be the number of Heads, and let Y be the number of Tails. I claim these two random events are not independent – demonstrate that this claim is correct.

Example

2: Here's our old friend  $S = \{10,11,12,13,14,15\}$ , with  $P(s) = \frac{1}{6} \forall s \in S$

Let  $X(s) = s \% 3$

Let  $Y(s) = 1$  if  $s \leq 12$   
 $= 2$  if  $s > 12$

s	X(s)	Y(s)
10	1	1
11	2	1
12	0	1
13	1	2
14	2	2
15	0	2

Claim: X and Y are independent random variables. To show this completely we could go through all six combinations of values for a and b (  $a \in \{0, 1, 2\}, b \in \{1, 2\}$  )

I'm not going to do them all, but we'll do one case:

Let a = 1 and b = 2.  $P(X=1) = \frac{1}{3}$ .  $P(Y=2) = \frac{1}{2}$

There is exactly one s for which  $X(s) = 1$  and  $Y(s) = 2$  (it's 13), so  $P(X=1 \text{ and } Y=2) = \frac{1}{6}$

Also,  $P(X=1) * P(Y=2) = \frac{1}{3} * \frac{1}{2} = \frac{1}{6}$

So the equation holds in this case. You can do the other 5 cases but I predict you will soon get bored – they all work out exactly the same way.

In practice we are often told that random variables are independent, or it may be obvious from the physical properties of the sample space. For example, S may consist of the outcomes of tossing two separate dice (say, a red one and a blue one) and we may accept that the two dice do not influence each other in any way. If X is a random variable based only on the value of the red die and Y is a random variable based only on the value of the blue die, then X and Y are independent.

This exercise shows how we can take advantage of knowing that two random variables are independent.

Exercise: Suppose  $X$  and  $Y$  are independent random variables, and we know two things:

$$P(X=23 \text{ and } Y=8) = 0.6$$

$$P(X=23) = 0.8$$

What is  $P(Y=8)$  ?

### Expectation

The concept of the **expected value** of a random variable is a formalization of our intuitive understanding of "average".

Suppose the ages of students in this class are all in the set  $\{16, 17, 18, 19, 20\}$ . Can we conclude that the average age in the class is 18? Of course not. To compute the average age we need to include all the instances of each value. If there are 4 people with age 16, 5 with age 17, 3 with age 18, 1 with age 19 and 1 with age 20, then the average is

$$\frac{4 \cdot 16 + 5 \cdot 17 + 3 \cdot 18 + 1 \cdot 19 + 1 \cdot 20}{14}$$

It works exactly the same way with random variables. Suppose  $X$  is a random variable that maps some sample space to the set  $\{1,2,3,4,5\}$  with the following probabilities:

$X(s)$	P
1	$\frac{1}{2}$
2	$\frac{1}{5}$
3	$\frac{1}{10}$
4	$\frac{1}{10}$
5	$\frac{1}{10}$

Then the **expected value** of  $X$  is given by

$$E(X) = \frac{\frac{1}{2} \cdot 1 + \frac{1}{5} \cdot 2 + \frac{1}{10} \cdot 3 + \frac{1}{10} \cdot 4 + \frac{1}{10} \cdot 5}{1} = \frac{1}{2} \cdot 1 + \frac{1}{5} \cdot 2 + \frac{1}{10} \cdot 3 + \frac{1}{10} \cdot 4 + \frac{1}{10} \cdot 5$$

where the denominator is 1 because it is just the sum of all the probabilities.

So in general, the expected value of a random variable  $X$  is

$$E(X) = \sum_{v \in V} P(X = v) \cdot v$$

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## Linearity of Expectation

Suppose  $X$  and  $Y$  are random variables defined on the same sample space  $(S,P)$

Define a new random variable  $Z$  on  $(S,P)$  as  $Z(s) = X(s) + Y(s)$

(remember,  $X$  and  $Y$  are functions, so we are just defining a new function by adding the previous ones together)

Theorem: If  $Z(s) = X(s) + Y(s)$  then  $E(Z) = E(X) + E(Y)$

Proof:

$$\begin{aligned} E(Z) &= \sum Z(s) \cdot P(s) \\ &= \sum (X(s) + Y(s)) \cdot P(s) \\ &= \sum X(s) \cdot P(s) + \sum Y(s) \cdot P(s) \\ &= E(X) + E(Y) \end{aligned}$$

More fun facts about expected values of random variables (we didn't cover these facts in class, but you should have no trouble verifying them!)

Let  $X$  and  $Y$  be random variables. Let  $c$  and  $d$  be any constant numbers. Then

$$E(X + c) = E(X) + c$$

$$E(c \cdot X) = c \cdot E(X)$$

$$E(c \cdot X + d \cdot Y) = c \cdot E(X) + d \cdot E(Y)$$

## Expectation of X\*Y

Let  $X$  and  $Y$  be random variables defined on sample space  $(S,P)$  and define  $Z(s) = X(s)*Y(s)$

Can we show that  $E(Z) = E(X)*E(Y)$  ? Not necessarily!

It turns out that if  $X$  and  $Y$  are **independent** random variables, then  $E(X*Y) = E(X)*E(Y)$

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The following material is optional – I won't test you on it, but it is useful information so I am including it in these notes in hopes that you will find it useful some day.

The text gives an equivalent formulation for  $E(X)$ : 
$$E(X) = \sum_{s \in S} P(s) \cdot X(s)$$

To see that these two formulas give the same value, remember that

$$P(X = v) = \sum_{s_i \text{ where } X(s_i)=v} P(s_i)$$

so we can write 
$$E(X) = \sum_{v \in V} \left( \sum_{s_i \text{ where } X(s_i)=v} P(s_i) \right) \cdot v$$

and in this, since we are specifying that  $X(s_i) = v$  in each term, we can replace  $v$  in the expression by  $X(s_i)$ , giving

$$E(X) = \sum_{v \in V} \left( \sum_{s_i \text{ where } X(s_i)=v} P(s_i) \right) \cdot X(s_i)$$

which we can rewrite as

$$E(X) = \sum_{v \in V} \left( \sum_{s_i \text{ where } X(s_i)=v} P(s_i) \cdot X(s_i) \right)$$



Well this is looking worse and worse. But if we think about what is happening here, we can see that we are actually just taking all terms of the form  $P(s_i) \cdot X(s_i)$  and grouping them together by their  $X(s_i)$  values. We don't change the sum by "ungrouping" them, so we end up with

$E(X) = \sum_{s_i \in S} P(s_i) \cdot X(s_i)$  and if we now discard the irrelevant subscripts on the  $s_i$  values – which were only introduced to clarify the equivalences – we end up with

$E(X) = \sum_{s \in S} P(s) \cdot X(s)$  ... as claimed.

You should make sure you understand why these are equivalent.

The text *also* gives this formulation  $E(X) = \sum_{a \in \mathbb{R}} a \cdot P(X = a)$  . If you are interested, the text gives a complete proof that this is equivalent to the previous formulation.

## Expectation of $X*Y$

Let  $X$  and  $Y$  be random variables defined on sample space  $(S,P)$  and define  $Z(s) = X(s)*Y(s)$

Can we show that  $E(Z) = E(X)*E(Y)$  ? Not necessarily!

It turns out that if  $X$  and  $Y$  are **independent** random variables, then  $E(X*Y) = E(X)*E(Y)$