QUEEN’S UNIVERSITY
FACULTY OF ARTS AND SCIENCE
DEPARTMENT OF COMPUTING AND INFORMATION SCIENCE

CISC-204*
Logic for Computer Scientists

TEST 2
February 17, 2006

Professors Robin Dawes (Section A) and Janice Glasgow (Section B)

Please write your answer to each question only in the box marked Answer.
No questions will be answered by the instructors during the exam,
This is a closed-book exam. No computers or calculators are allowed.
If you are unsure of what is wanted for a particular question,
make a reasonable assumption and write this at the beginning of your answer.

NAME: ____________________________  SECTION: __________

STUDENT NUMBER: ____________

FOR INSTRUCTOR’S USE ONLY

Question 1: _____ / 10

Question 2: _____ / 10

Question 3: _____ / 10

Question 4: _____ / 10

TOTAL: _____ / 40
Question 1: [10 marks]

Express the following sentences in predicate logic using the following predicate and constant definitions:

m: Mary
l: logic
Studies(x,y): x studies y
Course(x): x is a course
Comp(x): x is a computing student
Eng(x): x is an engineering student
Happy(x): x is happy

\textbf{Answer:}

(a) Mary is a computing student who studies logic.
\[\text{Comp}(m) \land \text{Studies}(m,l)\]

(b) Some, but not all, computer students study logic.
\[\exists x(\text{Comp}(x) \land \text{Study}(x,l)) \land \neg \forall x(\text{Comp}(x) \rightarrow S(x,l))\]

(c) Every computing student studies at least one course.
\[\forall x(\text{Comp}(x) \rightarrow \exists y(\text{Course}(y) \land \text{Studies}(x,y)))\]

(d) No computing student is unhappy.
\[\forall x(\text{Comp}(x) \rightarrow \neg \text{Happy}(x)) \text{ or } \neg \exists x(\text{Comp}(x) \land \neg \text{Happy}(x))\]

(e) All courses are studied by either some engineering students or some computing students.
\[\forall x(\text{Course}(x) \rightarrow \exists y((\text{Eng}(y) \lor \text{Comp}(y)) \land \text{study}(y,x)))\]
Question 2: [10 marks]

Prove the validity of the sequent

$$\forall x (P(x) \rightarrow Q(x)), \exists x (P(x) \land R(x)) \vdash \exists x (R(x) \land Q(x))$$

**Answer:**

1. $\forall x (P(x) \rightarrow Q(x))$ premise
2. $\exists x (P(x) \land R(x))$ premise

   $\frac{x_0}{3. \quad P(x_0) \land R(x_0)}$ assumption
   $4. \quad P(x_0)$ \quad $\land$ elimination, 3
   $5. \quad R(x_0)$ \quad $\land$ elimination, 3
   $6. \quad P(x_0) \rightarrow Q(x_0)$ $\forall$ elimination, 1
   $7. \quad Q(x_0)$ \quad $\rightarrow$ elimination, 4,6
   $8. \quad R(x_0) \land Q(x_0)$ $\land$ introduction, 5,7
   $9. \quad \exists x (R(x) \land Q(x))$ $\exists$ introduction 8

10. $\exists x (R(x) \land Q(x))$ $\exists$ elimination, 3-9
Question 3: [10 marks]

Let $\phi$ be the formula:

$$\forall x \forall y ((P(f(x,y)) \lor Q(y)) \rightarrow \forall z R(x,z))$$

Answer:

a) Draw the parse tree for the formula

b) Identify all bound and free variables within $\phi$

All variables are bound.

c) Let $\psi$ be the formula 

$$(P(f(x,y)) \lor Q(y)) \rightarrow \forall z R(x,z))$$

Compute the formula $\psi[f(a,b,z)/y]$

$$(P(f(x,f(a,b,z))) \lor Q(f(a,b,z)) \rightarrow \forall z R(x,z))$$

d) Which of $a$, $f(x)$ and $g(b,z)$ are free for $y$ in $\phi$.

They are all free except for $f(x)$ since $x$ would become bound by the universal quantifier $\forall x$. 
**Question 4: [10 marks]**

Give a formal proof of the following valid argument. Justify each step of the proof:

Premises:
1. \( \forall x (P(x) \rightarrow S(x)) \)
2. \( \forall x R(x) \)
3. \( \forall x (R(x) \rightarrow (P(x) \lor S(x))) \)

Conclusion:
\( \forall x (S(x)) \)

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**Answer:**

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<th>Step</th>
<th>Description</th>
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<td>1</td>
<td>( \forall x (P(x) \rightarrow S(x)) )</td>
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<td>2</td>
<td>( \forall x R(x) )</td>
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<td>3</td>
<td>( \forall x (R(x) \rightarrow (P(x) \lor S(x))) )</td>
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<td>12</td>
<td>( \forall x S(x) )</td>
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