QUEEN'S UNIVERSITY
FACULTY OF ARTS AND SCIENCE
DEPARTMENT OF COMPUTING AND INFORMATION SCIENCE

CISC-204
Logic for Computer Scientists

TEST 4
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Please write your answer to each question only in the box marked Answer.
No questions will be answered by the instructors during the exam.
This is a closed-book exam. No computers or calculators are allowed.
If you are unsure of what is wanted for a particular question,
make a reasonable assumption and write this at the beginning of your answer.

NAME: ____________________________   SECTION: ______________

STUDENT NUMBER: ____________________

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Question 1: ______ / 10

Question 2: ______ / 10

Question 3: ______ / 10

Question 4: ______ / 10

TOTAL: ______ / 40
Question 1: [10 marks]

Using the definition attached at the end of the quiz, prove the following logical equivalence:

\[ \phi W \psi \equiv \phi U \psi \lor G \phi \]

Answer:

\[ \phi W \psi \]

either \[ \Phi \rightarrow \ldots \rightarrow \Theta \]

all \( \pi^i \models \phi \) \< identical \> \[ \Theta \rightarrow \ldots \rightarrow \Theta \rightarrow \psi \rightarrow \]

all \( i \models \phi \)

\[ \Phi U \psi \lor G \phi \]

identical

\[ \Theta \rightarrow \ldots \rightarrow \Theta \rightarrow \psi \rightarrow \]

all \( i \models \phi \)

the set of paths that satisfy \( \phi W \psi \)

is identical to the set of paths that satisfy \( \Phi U \psi \lor G \phi \)

so the equivalence holds.
Question 2: [10 marks]

Consider the model $\mathcal{M}$ described below. For each of the formulas $\phi$ listed, find a path from the initial state $s_0$ that satisfies $\phi$ and determine whether $\mathcal{M}, s_0 \models \phi$. Justify your answer.

Answer:

(a) $p \rightarrow XX\neg q$

$\Gamma: s_0 \rightarrow s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow \ldots$ satisfies $\phi$

$\mathcal{M}, s_0 \not\models \phi$ since the path $\Gamma = s_0 \rightarrow s_3 \rightarrow s_3 \rightarrow \ldots$ does not satisfy $\phi$

(b) $G(p \rightarrow Xq)$

$\Gamma: s_0 \rightarrow s_1 \rightarrow s_2 \rightarrow \ldots$ satisfies $\phi$

$\mathcal{M}, s_0 \models \phi$ since the only state where $p$ holds is $s_0$, and $q$ holds in all successors to $s_0$

(c) $FGq$

$\Gamma: s_0 \rightarrow s_2 \rightarrow s_3 \rightarrow \ldots$ satisfies $\phi$

$\mathcal{M}, s_0 \not\models \phi$ since the path $\Gamma = s_0 \rightarrow s_3 \rightarrow s_3 \rightarrow s_3 \rightarrow \ldots$

has infinitely many states satisfying $\phi$
Question 3: [10 marks]

Assume you have a system with three processes, where each process can be in one of two states, \textit{off} or \textit{on}, and there are transition relations \textit{on} \rightarrow \textit{off} and \textit{off} \rightarrow \textit{on} for each process. For any state transition, exactly one process can change its state.

**Answer:**

Define the atomic propositions needed to model this system.

$$\begin{align*}
\rho_1 &= \text{true if process 1 is on, false if process 1 is off,} \\
\rho_2 &= \ldots \\
\rho_3 &= \ldots
\end{align*}$$

Express each of the following properties in linear-time temporal logic using the propositions defined above.

- All processes are \textit{off} in the initial state for the model.

  Assume $S_0$ is the initial state

  $$S_0 \models \neg \rho_1 \land \neg \rho_2 \land \neg \rho_3$$

- It is not possible for all three processes to be in their \textit{on} state at any given time.

  $$\text{G} \neg \left( \rho_1 \land \rho_2 \land \rho_3 \right)$$

- Whenever process 1 is in its \textit{on} state, then in the next state process 2 must be \textit{off}.

  $$\text{G} \left( \rho_1 \rightarrow \neg \rho_2 \right)$$
Question 4: [10 marks]

Using a transition diagram, construct a model for the system that satisfies the properties expressed in Question 3.

Answer:
Definition: Let $\mathcal{M} = \{S, \rightarrow, L\}$ be a model and $\pi = s_1 \rightarrow \ldots$ be a path in $\mathcal{M}$. Whether path $\pi$ satisfies an LTL formula is defined by the satisfactrion $\models$ relation as follows:

1. $\pi \models \top$

2. $\pi \not\models \bot$

3. $\pi \models p$ iff $p \in L(s_1)$

4. $\pi \models \neg \phi$ iff $\pi \not\models \phi$

5. $\pi \models \phi \land \psi$ iff $\pi \models \phi$ and $\pi \models \psi$

6. $\pi \models \phi \lor \psi$ iff $\pi \models \phi$ or $\pi \models \psi$

7. $\pi \models \phi \rightarrow \psi$ iff $\pi \models \psi$ whenever $\pi \models \phi$

8. $\pi \models X\phi$ iff $\pi^2 \models \phi$

9. $\pi \models G\phi$ iff $\pi^i \models \phi$ for all $i \geq 1$

10. $\pi \models F\phi$ iff $\pi^i \models \phi$ for some $i \geq 1$

11. $\pi \models \phi U \psi$ iff there is some $i \geq 1$ such that $\pi^i \models \psi$ and for all $j = 1, \ldots, i - 1$ we have $\pi^j \models \phi$

12. $\pi \models \phi W \psi$ iff either there is some $i \geq 1$ such that $\pi^i \models \psi$ and for all $j = 1, \ldots, i - 1$ we have $\pi^j \models \phi$; or for all $k \geq 1$ we have $\pi^k \models \phi$

13. $\pi \models \phi R \psi$ iff either there is some $i \geq 1$ such that $\pi^i \models \phi$ and for all $j = 1, \ldots, i$ we have $\pi^j \models \psi$; or for all $k \geq 1$ we have $\pi^k \models \psi$. 