CISC 204 Class 2

Proof Rules for Conjunction and Double Negation

Text Correspondence: pp. 6–8

Main Concepts:

- $\wedge i$*: conjunction introduction*
- $\wedge e_1$,$\wedge e_2$ *: conjunction elimination*
- *Operator: logical connective*

The first proof rules that we will explore, for natural deduction, are the rules for conjunction and double negation.

2.1 Elementary Proof Rules: Conjunction

The first rules are for conjunction, which in English is imperfectly captured by the word "and". The idea is that if two formulas are each true, therefore the conjunction of the formulas is true; also, if a conjunction of formulas is true, therefore each conjunct is true.

Proof Rule: **conjunction**-*introduction*: $\wedge$i

$$\frac{\phi \quad \psi}{\phi \wedge \psi}\ \wedge\text{i}$$

Proof Rule: **conjunction**-*elimination*: $\wedge$e

$$\frac{\phi \wedge \psi}{\phi}\ \wedge\text{e}_1$$

or

$$\frac{\phi \wedge \psi}{\psi}\ \wedge\text{e}_2$$

We can use the proof rules for conjunction to show that the following sequents are valid:

$$p \wedge q, r \vdash q \wedge r$$
$$(p \wedge q) \wedge r, s \wedge t \vdash q \wedge s$$

As a self-study problem, express the following declarative sentence in propositional logic. You may define your own symbols for propositional atoms.

> If you study hard and do all of the assignments then you will get a good mark in the class, but if you do not study or do not do all of the assignments then you will not get a good mark in the class.

2.2 Elementary Proof Rules: Double Negation

There are two proof rules for *double negation*. The first double-negation rule is that if a given formula is not-untrue, therefore it is true; the second double-negation rule is that if a formula is true, therefore it is not-untrue.

Proof Rule: **double negation**-*elimination*: $\neg\neg$ e

$$\frac{\neg\neg\,\phi}{\phi}\ \neg\neg\,\text{e}$$

Proof Rule, derived: **double negation**-*introduction*: $\neg\neg$ i

$$\frac{\phi}{\neg\neg\,\phi}\ \neg\neg\,\text{i}$$

Self-Study Problems

Express the following in propositional logic:

- Today it will rain or shine but not both.
- My sister wants a black and white cat.
- No one will pass the course who does not study.
- John and Mary both went to the show; John liked the show but Mary did not.

Using the negation rules and the conjunction rules, prove that this more complicated sequent is valid:

$$p, \neg\neg\,(q \wedge r) \vdash \neg\neg\, p \wedge r \qquad \textit{H\&R, p. 8}$$

Self-Study Sequent 2.1: $p \wedge q, r \vdash q \wedge r$

Self-Study Sequent 2.2: $(p \wedge q) \wedge r, s \wedge t \vdash q \wedge s$

Sample proofs for these self-study sequents are on the next page of these notes.

Self-Study Sequent 2.1: $p \wedge q, r \vdash q \wedge r$

1	$p \wedge q$	premise
2	r	premise
3	q	$\wedge e_2$ 1
4	$q \wedge r$	$\wedge i$ 3, 2

Self-Study Sequent 2.2: $(p \wedge q) \wedge r, s \wedge t \vdash q \wedge s$

1	$(p \wedge q) \wedge r$	premise
2	$s \wedge t$	premise
3	$p \wedge q$	$\wedge e_1$ 1
4	q	$\wedge e_2$ 3
5	s	$\wedge e_1$ 2
6	$q \wedge s$	$\wedge i$ 4, 5