CISC 204 Class 4

Proof Rules for Negation and Implication Introduction

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Main Concepts:

- $\neg e$*: negation elimination*
- $\bot e$*: bottom elimination*
- *Assumption box: notation to indicate the use of an assumption in a logical proof*
- $\rightarrow i$*: implication introduction*

This class explores two proof rules that use the concept of *contradiction*. In natural deduction, as in most systems for symbolic logic, the word "contradiction" has a specific technical meaning.

We will use the "bottom" symbol, $\bot$, to represent a contradiction. When we explore the semantics of propositional logic, we will see that this symbol must always have a valuation of "false". For the purposes of proofs in propositional logic, the bottom symbol will represent a stage in a proof where a contradiction has arisen.

Definition: Contradiction

A **contradiction**, which we write as $\bot$, is an expression of the form

$$\phi \wedge \neg\,\phi \quad \text{or}\ \neg\,\phi \wedge \phi$$

For example, $(p \vee q) \wedge \neg\,(p \vee q)$ is a contradiction.

Any formula can be can be derived from a contradiction:

$$p \wedge \neg\, p \vdash \phi$$

This gives us two more rules of deduction for negation.

4.1 Negation Elimination

Our first proof rule for negation allows us to deduce a contradiction. The proof rule requires the presence of a proposition ϕ in a preceding line of a valid proof, and also the presence of a proposition $\neg\phi$ in some other preceding line of a valid proof. From the presence of these two propositions, we can deduce – that is, we can prove – that there is a contradiction in our proof.

Proof Rule: **negation**-*elimination*, $\neg$ e

$$\frac{\phi \quad \neg\phi}{\bot} \neg \mathrm{e}$$

It is reasonable to ask whether this rule could equally well be named “bottom introduction”. The answer is, simply, that it could be so named but for historical reasons it is called “negation elimination”.

4.2 Bottom Elimination

The second proof rule for this class is a formalization of the idea that, from a contradiction, we can deduce anything. In natural deduction, the proof rules is that if there is a contradiction as a preceding line of a valid proof, then any proposition ϕ can be deduced or proved.

Proof Rule: **bottom**-*elimination*, $\bot$e

$$\frac{\bot}{\phi} \bot\mathrm{e}$$

These two rules are often used in combination in a logical proof. A typical strategy is: find contradictory propositions from premises or assumptions; deduce a a contradiction; and deduce a goal proposition, such as the conclusion of a sequent.

Here is an example sequent that can be proved using these two rules.

Self-Study Sequent 4.1: $p \to r, p \wedge \neg q, \neg r \vdash q$

A sample proof for this sequent is on the next page of these notes.

Self-Study Sequent 4.1: $p \to r, p \wedge \neg q, \neg r \vdash q$

1	$p \to r$	premise
2	$p \wedge \neg q$	premise
3	$\neg r$	premise
4	p	$\wedge e_1$ 2
5	r	$\to$e 4, 1
6	$\perp$	$\neg$e 5, 3
7	q	$\perp$i 6

4.3 Implication Introduction

Another mode of reasoning is sometimes called "what-if". Suppose, for example, that we assume that a proposition is *tentatively* true "for the purpose of this argument" and deduce a *goal* proposition; the truth of the assumption materially implies the truth of the goal.

The proof rule in natural deduction that codifies this way of reasoning is *implication introduction*. This powerful rule of deduction is one that we will use often in this course.

Proof Rule: **implication**-*introduction*: $\to$i

$$\frac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \psi\end{array}}}{\phi \to \psi}\ \to\text{i}$$

Here is how we use this rule:

- Temporarily assume that ϕ is true. If you can prove that ψ is true using this assumption, you can prove $\phi \to \psi$.
- Keep track of assumptions by putting the part of proof that has the assumption inside a box. This box denotes the scope of the temporary assumption.
- You cannot use any lines within a box once you have left it.
- All assumptions must be "released" before a proof is complete.

As an example, we can prove the sequent: $p \to q \vdash \neg\, q \to \neg\, p$

1	$p \to q$	premise
2	$\neg\, q$	assumption
3	$\neg\, p$	MT 1, 2
4	$\neg\, q \to \neg\, p$	$\to$i 2–3

As another example, we can prove the sequent: $p \to q, q \to r \vdash p \to r$

1	$p \to q$	premise
2	$q \to r$	premise
3	p	assumption
4	q	$\to$e 3, 1
5	r	$\to$e 4, 2
6	$p \to r$	$\to$i 3–5

Here are two sequents for self-study. Example proofs are on the next page of these notes.

Self-Study Sequent 4.2: $p \vdash (p \to q) \to q$

Self-Study Sequent 4.3: $\neg q \to \neg p \vdash p \to q$

Self-Study Sequent 4.2: $p \vdash (p \to q) \to q$

1	p	premise
2	$p \to q$	assumption
3	q	$\to$e 1, 2
4	$(p \to q) \to q$	$\to$i 2–3

Self-Study Sequent 4.3: $\neg q \to \neg p \vdash p \to q$

1	$\neg q \to \neg p$	premise
2	p	assumption
3	$\neg\neg p$	$\neg\neg$i 2
4	$\neg\neg q$	MT 1, 3
5	q	$\neg\neg$ e 4
6	$p \to q$	$\to$i 2–5

Many of the proofs in this course involve *nested* assumption boxes, in which an "outer" assumption box has one or more "inner" assumption boxes. This nesting can be as deep as required. There is no limit, but: the proof must be finite in length.

4.4 Theorems

We can now introduce a new term, which is a *theorem*. That a proposition ϕ is a theorem means that ϕ is the conclusion of a valid sequent that has no premises.

Definition: A logical formula ϕ that has a valid sequent $\vdash \phi$ is a *theorem*.

A simple example of a theorem is: $\vdash p \to p$

An example of a theorem that can be proved using nested assumption boxes is: $\vdash p \to (q \to p)$

Proof of: $\vdash p \to (q \to p)$

1	p	assumption
2	q	assumption
3	p	copy 1
4	$q \to p$	$\to$ i 2–3
5	$p \to (q \to p)$	$\to$ i 1–4