CISC 204 Class 16

Proof Rules For Universal Quantifiers

Text Correspondence: pp. 109–112

Main Concepts:

- *Universal Elimination: replacing a universal quantifier with a "fresh" variable*
- *Universal Introduction: replacing an assumed "fresh" variable with a universal quantifier*

Formal reasoning, or proofs, of sequents involving quantifiers begins with some simple observations. The first observation is that, if a formula is true for *all* values of a variable, then it is true for *some specific* value of the variable. We are familiar with this concept from elementary algebra, where a variable in an equation can be replaced with a numerical value; here, we extend the concept to a logical formula.

16.1 Universal Elimination

Consider a formula ϕ that has a variable x, and a term t that has t free for x in ϕ. If the quantified formula

$$\forall x\, \phi$$

is true, then the substitution of t for x, which is

$$\phi[t/x]$$

must also be true. We must be careful here: the formula ϕ has t free for x, but the formula $\forall x\, \phi$ has x in its scope. We are saying that a universal formula can be "instantiated" with a new term, within the scope of the quantifier.

Proof Rule: Universal Elimination, $\forall x$ e

$$\frac{\forall x\, \phi}{\phi[t/x]}\ \forall x\ \mathrm{e}$$

Key Concept: The $\forall x$ e rule uses "forward" logical reasoning. When we invoke this rule, we substitute a "fresh" variable for the variable that is bound by the $\forall$ quantifier. The substituted formula is the result of applying the rule to the quantified formula.

The use of $\forall x$ e can be illustrated using a simple sequent.

Example

Consider the situation where we know that a predicate $P(\cdot)$ is true of a specific value a; technically, a is a constant function that evaluates to a value. We say that a has property P.

Suppose further that, for any value of a variable x, if x has property P then x does not have property Q. We naturally want to conclude that a does not have property Q. The sequent for this example is

$$P(a), \forall x\,(P(x) \rightarrow \neg\, Q(x)) \vdash \neg\, Q(a) \tag{16.1}$$

The proof strategy is that we can replace the bound occurrence of x in the second premise with the definite term a. We will annotate our proof in more detail than is recommended by the textbook, so that our proof of Argument 16.1 is

1	$P(a)$	premise	
2	$\forall x\,(P(x) \rightarrow \neg\, Q(x))$	premise	
3	$P(a) \rightarrow \neg\, Q(a)$	$\forall x$ e 2	using $\phi[a/x]$
4	$\neg\, Q(a)$	$\rightarrow$ e 3, 1	

16.2 Universal Introduction

The next rule of predicate logic is based on the idea that, if we can prove something for a newly introduced variable, then because the proof does not depend on any specific value or binding of the new variable then the proof applies to *all* values of the variable.

More formally, suppose that we introduce a "fresh" variable x_0 that is free for x in a formula ϕ. When we have proved ϕ using the variable x_0, we have performed an extension of the propositional logic rule $\rightarrow$i. The logic, in a proof, would be

$$\frac{\boxed{\begin{array}{ll} x_0 & \\ & \vdots \\ & \phi[x_0/x] \end{array}}}{\forall x\,\phi}\ \forall x \text{ i}$$

where the last line is how we state that the "fresh" variable x_0 can be substituted. From this, we want to be able to conclude that ϕ can be universally quantified.

Before defining the new proof rule, we can consider an example of how we might want to use such a rule.

Example

From propositional logic, we know that $p \vee \neg p$ is a tautology. A straightforward extension into predicate logic would be, for a variable x and a property Q, that for all x either x has the property Q or it does not; in symbols this is $\forall(Q(x) \vee Q(x))$.

Our general way of reasoning is that, if we use the tautology so that it has a "new" free variable x_0 in it, then we can universally quantify the tautology. We want to be able to introduce the new variable and draw our conclusion from it. The argument we want to use has no premises; the sequent has only a conclusion

$$\vdash \forall x(Q(x) \vee Q(x))$$

Our specific line of reasoning is this: from the Law of Excluded Middle, with a new variable x_0, we can assert $Q(x_0) \vee \neg Q(x_0)$. Because we have not specified a specific value for x_0, we want to be able to conclude that the disjunction is true for all values of x_0. The form of the argument we want to use is

$$\frac{\boxed{\begin{array}{lll} x_0 & & \\ & Q(x_0) \vee Q(x_0) & \text{LEM} \end{array}}}{\forall x(Q(x) \vee Q(x))}\ \forall x\ \text{i}$$

To be able to use this kind of argument, we need a new proof rule.

Proof Rule: Universal Introduction, $\forall x$ i

$$\frac{\boxed{\begin{array}{ll} x_0 & \\ & \vdots \\ & \phi[x_0/x] \end{array}}}{\forall x\, \phi}\ \forall x\ \text{i}$$

One way to express this rule in English is: introducing a new variable x_0, if we can prove a formula ϕ that has the new variable x_0 substituted into it, then we can conclude that ϕ is valid when universally quantified.

Key Concept: The $\forall x$ i rule uses "backward" logical reasoning. When we invoke this rule, we introduce a "fresh" variable as an assumption inside the box. We substitute this fresh variable into the quantified formula that we are trying to prove, and write the substituted formula as the last line in the box.

Example

Consider an example where we assert that every x that has property P also has property Q. We must be careful in translating this English into symbols, so that we get

$$\forall x \, (P(x) \to Q(x)) \tag{16.2}$$

We may further assert that every x has property P. From this, we want to conclude that every x has property Q. Combining Argument 16.2 with translations of the second assertion and the conclusion, we want to prove the sequent

$$\forall x \, (P(x) \to Q(x)), \forall x \, P(x) \vdash \forall x \, Q(x) \tag{16.3}$$

After writing the premises, we will assume the use of a new variable z. Using universal elimination, the premise of Argument 16.2 allows us to deduce that $P(z) \to Q(z)$. Using universal elimination again, the second premise $\forall x \, P(x)$ allows us to deduce $P(z)$. Applying the rule Modus Ponens, which is implication elimination, we can deduce $Q(z)$.

Because the variable z is new and is also free, we can use universal introduction to deduce that $\forall x \, Q(x)$. Our proof is:

1		$\forall \, x(P(x) \to Q(x))$	premise
2		$\forall x \, P(x)$	premise
3	z		
4		$P(z) \to Q(z)$	$\forall x$ e 1
5		$P(z)$	$\forall x$ e 2
6		$Q(z)$	$\to$ e 4, 5
7		$\forall x \, Q(x)$	$\forall x$ i 3–6 where ϕ is $Q(x)$

In English, one way to read $\phi[t/x]$ is: ϕ in which every free occurrence of x is replaced by t.

In this example, we introduced a variable z that led to the formula of Line 6; because this z does not have a specified value, the formula ϕ is true for *all* values, so we can universally quantify the formula ϕ.

Exercise

Students should now be able to modify the above example to prove the validity of the sequent

$$\forall x\,(P(x) \to Q(x)) \vdash \forall x\,P(x) \to \forall x\,Q(x) \tag{16.4}$$

Hint: consider the propositional logic rule $\to$ i.