CISC 204 Class 21

Mixing Quantifiers With Propositional Logic

Text Correspondence: pp. 120–122

Main Concepts:

- *Predicate logic is an extension of propositional logic*

We should now be prepared to integrate our understanding of the rules of predicate logic with our previous understanding of propositional logic. To see how far we have come, let us try to prove Syllogism 13.1, which we used when we introduced the concepts of predicates and quantifiers.

Example "All humans are mortal and Socrates is human, so Socrates is mortal."

This syllogism, translated into symbols, is the sequent

$$\forall x\,(H(x) \to M(x))\,,\ \exists x\,(S(x) \wedge H(x)) \vdash \exists x\,(S(x) \wedge M(x)) \tag{21.1}$$

By inspection, because quantifiers are present, we can see it is likely that we will need assumption boxes. We also see an implication in the premises, so we are prepared to use some of our reasoning from propositional logic.

A first step is to introduce a "fresh" variable z in place of the existential quantifier of the second premise; this would give us $S(z) \to H(z)$. We can substitute into the universal quantifier of the first premise using *the same* variable z; this would give us $H(z) \to M(z)$. Propositional reasoning can be used to deduce $S(z) \to M(z)$ by implication introduction. We can then existentially quantify over the "fresh" variable z to get a formula that is like the conclusion; finally, we can use existential elimination over the "fresh" variable and we will have deduced the conclusion.

Our proof of Sequent 21.1, which follows this line of reasoning, is:

1		$\forall x\,(H(x) \to M(x))$	premise
2		$\exists x\,(S(x) \wedge H(x))$	premise
3	z	$S(z) \wedge H(z)$	assumption
4		$H(z) \to M(z)$	$\forall x$ e, 1
5		$S(z)$	$\wedge$ e_1 3
6		$H(z)$	$\wedge$ e_2 3
7		$M(z)$	$\to$ e 4,6
8		$S(z) \wedge M(z)$	$\to$ i 5–7
9		$\exists x\,(S(x) \wedge M(x))$	$\exists x$ i 8
10		$\exists x\,(S(x) \wedge M(x))$	$\exists x$ e 2, 3–9

In this proof we used simple propositional rules inside a predicate-related assumption box. This is a useful strategy for more complicated proofs.

Example “All humans are mortal and Greeks are human, so Greeks are mortal.”

This classical syllogism, translated into symbols, is the sequent

$$\forall x\,(H(x) \to M(x))\,,\ \forall x\,(G(x) \to H(x)) \vdash \forall x\,(G(x) \to M(x)) \tag{21.2}$$

After writing the premises and the conclusion, we see that we will need a “fresh” variable to deduce the conclusion; we can use z because that variable is free in each relevant formula. This requires a $\forall x$ i assumption box. The last line of the $\forall x$ i box is the implication $G(z) \to M(z)$, so we will need to open a $\to$ i box that has $G(z)$ as its assumption and that has $M(z)$ as its last line. Our reasoning so far is

1	$\forall x\,(H(x) \to M(x))$	premise
2	$\forall x\,(G(x) \to H(x))$	premise
3	z	
	$\vdots$	
	$G(z)$	assumption
	$\vdots$	
	$M(z)$	
	$G(x) \to M(x)$	$\to$ i ?–?
?	$\forall x\,(G(x) \to M(x))$	$\forall x$ i 3–?

To be able to use the proposition $G(z)$, we will need to substitute Line 2 with the “fresh” variable z. The missing element comes from Line 1; we can *also* use the “fresh” variable z. This second idea is semantically justified because if $H(x) \to M(x)$ is true of all x, then it is true of the particular variable indicated by z; the second idea is syntactically permitted by the $\forall x$ e rule.

Two applications of the $\to$ e rule completes our proof.

1	$\forall x\,(H(x) \to M(x))$	premise
2	$\forall x\,(G(x) \to H(x))$	premise
3	z	
4	$H(z) \to M(z)$	$\forall x$ e 1
5	$G(z) \to H(z)$	$\forall x$ e 2
6	$G(z)$	assumption
7	$H(z)$	$\to$ e 5, 6
8	$M(z)$	$\to$ e 4, 7
9	$G(x) \to M(x)$	$\to$ i 6–8
10	$\forall x\,(G(x) \to M(x))$	$\forall x$ i 3–9

This proof is an example of how to substitute one variable into two distinct formulas.

Example One textbook problem is the sequent

$$\forall x\,(\neg P(x) \vee Q(x)) \vdash \forall x\,(P(x) \rightarrow Q(x)) \tag{21.3}$$

We can prove this sequent by observing that the conclusion requires us to open a $\forall x$ i assumption box; we can use the variable z that is free in the premise and in the conclusion. This box ends with the formula $P(z) \rightarrow Q(z)$ so we will open a $\rightarrow$ i assumption box.

1	$\forall x\,(\neg P(x) \vee Q(x))$	premise
2	z	
3	$P(z)$	assumption
	$\vdots$	
?	$Q(z)$	$\vee$ e 4, 5–7, 8–8
?	$P(z) \rightarrow Q(z)$	$\rightarrow$ i 3–9
?	$\forall x\,(P(x) \rightarrow Q(x))$	$\forall x$ i 2–10

We can then substitute z into the premise, which produces the formula $\neg P(z) \vee Q(z)$. We have a choice about where we do this substitution: it can be immediately after Line 2, where we introduced the "fresh" variable z, or it can be inside the $\rightarrow$ i assumption box as Line 4. Here we will take the latter approach, understanding that some authors prefer the former.

Now we can think using forward logic. This is a disjunction so we can use disjunction elimination: one assumption box assumes the left disjunct $\neg P(z)$ and the other assumes the right disjunct $Q(z)$. The left disjunct leads directly to a contradiction, so we can deduce $Q(z)$ which is our goal. The right disjunct *is* our goal: we now have a proof.

1	$\forall x\,(\neg P(x) \vee Q(x))$	premise
2	z	
3	$P(z)$	assumption
4	$\neg P(z) \vee Q(z)$	$\forall x$ e 1
5	$\neg P(z)$	assumption
6	$\perp$	$\neg$ e 3,5
7	$Q(z)$	$\perp$ e
8	$Q(z)$	assumption
9	$Q(z)$	$\vee$ e 4, 5–7, 8–8
10	$P(z) \rightarrow Q(z)$	$\rightarrow$ i 3–9
11	$\forall x\,(P(x) \rightarrow Q(x))$	$\forall x$ i 2–10

A student may find it useful to complete the proof by substituting Line 1 immediately after Line 2, before the $\rightarrow$ i assumption box. The proof is nearly identical but doing this alternative substitution uses the "fresh" variable z differently.