CISC 204 Class 28

Satisfiability in Predicate Logic

Text Correspondence: pp. 130–131

Main Concepts:

- *Satisfiable formula: there is some model in which the formula evaluates to* **T**

Previously, we found a model and an environment that a formula ϕ *satisfies*. We now can provide a definition of satisfaction of a formula in predicate logic.

28.1 Satisfiability of a Formula

The concept of formula satisfaction is defined recursively. In English, the base case is: a predicate P is satisfied in a model means that, when we map its terms to the universe A using the environment l, the resulting values are in the set P. The base case needs to accommodate an n-ary predicate, and the predicate needs to be extensionally defined as a set.

The notation for the base case is that, for an interpretation $\mathcal{I}$, we represent an n-ary predicate $P(t_1, t_2, \ldots, t_n)$ being satisfied for terms $t_1, t_2, \ldots, t_n$ as

$$\mathcal{M} \models_l P(t_1, t_2, \ldots, t_n)$$

From predicates, we build up the concept of satisfaction in two ways. The first is, in parallel to propositional logic, we define satisfaction for the logical operators $\{\neg, \land, \lor, \rightarrow\}$. The second, specific to predicate logic, is that we define satisfaction for universal quantification $\forall x$ and for existential quantification $\exists x$.

Definition: Formula satisfaction in an interpretation

Given an interpretation $\mathcal{I}$, the satisfaction relation $\mathcal{M} \models_l \phi$ *holds for* ϕ means:

If ϕ is $P(t_1, t_2, \ldots, t_n)$, calculate the value of each term t_i using the mapping l to find the value $t_i = a_i \in A$; in each calculation, use the model of $f \in \mathcal{F}$ as $f^{\mathcal{M}}$. Then $\mathcal{M} \models_l \phi$ *holds* means that $(a_1, a_2, \ldots, a_n) \in P^{\mathcal{M}}$.

An equivalent statement is: $\mathcal{M} \models_l \phi$ evaluates to **T** if $(a_1, a_2, \ldots, a_n) \in P^{\mathcal{M}}$ and evaluates to **F** if $(a_1, a_2, \ldots, a_n) \notin P^{\mathcal{M}}$.

Otherwise, $\mathcal{M} \models_l \phi$ *holds* means that one of these cases evaluates to **T**:

Case	**Requirement**
$\mathcal{M} \models_l \neg\psi$ evaluates to **T**	$\mathcal{M} \models_l \psi$ does not evaluate to **T**
$\mathcal{M} \models_l \psi_1 \wedge \psi_2$ evaluates to **T**	$\mathcal{M} \models_l \psi_1$ evaluates to **T** and $\mathcal{M} \models_l \psi_2$ evaluates to **T**
$\mathcal{M} \models_l \psi_1 \vee \psi_2$ evaluates to **T**	$\mathcal{M} \models_l \psi_1$ evaluates to **T** or $\mathcal{M} \models_l \psi_2$ evaluates to **T**
$\mathcal{M} \models_l \psi_1 \to \psi_2$ evaluates to **T**	$\mathcal{M} \models_l \psi_2$ evaluates to **T** whenever $\mathcal{M} \models_l \psi_1$ evaluates to **T**
$\mathcal{M} \models_l \forall x\, \psi$ evaluates to **T**	$\mathcal{M} \models_{l[x \mapsto a]} \psi$ evaluates to **T** for all $a \in A$
$\mathcal{M} \models_l \exists x\, \psi$ evaluates to **T**	$\mathcal{M} \models_{l[x \mapsto a]} \psi$ evaluates to **T** for some $a \in A$

An important special situation for satisfaction occurs when a formula ϕ has no free variables. In such a situation, the formula ϕ either holds or does not hold in model $\mathcal{M}$ regardless of the choice of the environment l. This is because, having no free variables, the assignment of variables in the environment has no effect on the semantics of ϕ. To summarize this situation:

- If ϕ has no free variables, it is called a *sentence* in predicate logic;
- A sentence holds or does not hold in a model $\mathcal{M}$ independent of the choice of the environment l, so we can use the abbreviation $\mathcal{M} \models \phi$ for sentence ϕ

Example: Base-4 arithmetic

Our model of base-4 arithmetic had two predicates because $\mathcal{P} = \{P, Q\}$. We will consider three closely related formulas; a theorem, a sentence that is satisfied in the model, and a formula with a free variable.

Consider the English statement

Every number is even or odd

We can translate this in at least two distinct ways. The first is to say that every number is even or is not even. This gives us the sentence

$$\forall x\,(P(x) \vee \neg\, P(x)) \tag{28.1}$$

We can easily verify that Formula 28.1 is a theorem, and that it is satisfied in the model $\mathcal{M}$.

A second way of translating the English statement is to use both predicates, one for the even numbers and one for the odd numbers. This gives us the sentence

$$\forall x\,(P(x) \vee Q(x)) \tag{28.2}$$

To determine whether Formula 28.2 holds, we would have to apply the definition of $\models_l$ recursively. This recursion is only of depth 2, so it can be expanded in-line. The applications of the definition of satisfaction look like:

1. for each $a \in A$:
2.   assign x with value a
3.   evaluate $\vee$:
4.   evaluate $a \in P$
5.   evaluate $a \in Q$
6.   evaluates to **`T`** if Line 4 is **`T`** or Line 5 is **`T`**
7.   evaluates to **`T`** if every Line 6 evaluates to **`T`**

In Line 4 and Line 5, we have used the extensional definitions of the predicates. Line 6 is a reminder to us of the definition of "evaluates" for disjunction, and Line 7 is a reminder of universal quantification. For this model, the sentence in Formula 28.2 is satisfied.

Students should verify that Formula 28.2 does not hold for a model $\mathcal{M}'$ in which we use a modified predicate

$$Q' = \{1\}$$

The theorem of Formula 28.1 still holds under this alternate model $\mathcal{M}'$.

Next, consider the English statement

The number z is even or odd

We will use the two-predicate translation, parallel to Formula 28.2, so that we have the formula

$$P(z) \vee Q(z) \tag{28.3}$$

The ϕ of Formula 28.3 is not a sentence. Examining it, we see that the variable z is free in ϕ. The model $\mathcal{M}$ now needs an environment l for us to be able to determine whether or not Formula 28.3 holds in $\mathcal{M}$.

Let us consider two environments, l_1 and l_2, within this model $\mathcal{M}$ of base-4 arithmetic. Each of these is an assignment of the variable z to a value in A. We can pick

$$l_1 \stackrel{\text{def}}{=} \{(z, 0)\} \tag{28.4}$$

$$l_2 \stackrel{\text{def}}{=} \{(z, 1)\} \tag{28.5}$$

When we apply the definition of satisfaction to Formula 28.3, we get

1. In environment l_1, assign z with value $l_1(z) = 0$
2. evaluate $\vee$:
3.   evaluate $0 \in P$
4.   evaluate $0 \in Q$
5.   evaluates to **T** if Line 3 is **T** or Line 4 is **T**

As we expected from the English statement, Formula 28.3 is satisfied in Model $\mathcal{M}$ with environment l_1.

Students should verify that, using environment l_2, Formula 28.3 is also satisfied.

A simple but useful exercise is to find a formula that is satisfied in environment l_1 and is not satisfied in environment l_2.