CISC 204 Class 32

Completeness in Predicate Logic

Text Correspondence: pp. 131–136

Main Concepts:

- *Complete logical theory: every semantic entailment from zero premises is a theorem*

32.1 Completenes of Predicate Logic

An important result, originally proved by Kurt Gödel, is the *Completeness Theorem*. It can be written concisely as

$$\text{if } \Gamma \models \psi, \text{ then } \Gamma \vdash \psi$$

In propositional logic, we could prove this by enumerating the 2^k valuations of the formula ψ; this is tedious but can be automated. In predicate logic, the Completeness Theorem can be proved by reasoning over the logical operators and quantifiers.

We will not prove these theorems, but we will use their concepts when we turn to considering how to find a proof for a valid formula (hint: this cannot be done in general).

Example: Consider the formula ϕ that is $\exists x\,(P(x) \to Q(x))$

Let us show that both ϕ and $\neg\,\phi$ are satisfiable.

To show that ϕ is satisfiable, we need to devise an interpretation in which ϕ evaluates to **T**. That is, we need to devise:

- a universe of discourse A
- a model $\mathcal{M}$ with predicates $\mathcal{P} = \{P^{\mathcal{M}}, Q^{\mathcal{M}}\}$
- at least one environment $x \mapsto a$

Because the specific form of ϕ is existential quantification, meaning that it has the form $\exists x\,\psi$ where x is a free variable in ψ, we need to find just one assignment of x that satisfies ψ.

Let us try by starting with the simple universe $A = \{0, 1\}$. We need to have one assignment of x that "works", so let us pick $x \mapsto 0$. Substituting this assignment into ψ, we need to find at least one interpretation $P^{\mathcal{M}}$ and $Q^{\mathcal{M}}$ such that

$$P^{\mathcal{M}}(0) \to Q^{\mathcal{M}}(0)$$

evaluates to **T**. We can do this if $P^{\mathcal{M}}(0)$ evaluates to **F**; an easy way to do this is to select the predicate

$$P^{\mathcal{M}} \stackrel{\text{def}}{=} \{1\}$$

In this model and environment, the implication ψ evaluates to **T** so we can pick any predicate $Q^{\mathcal{M}}$ whatsoever. Let us pick one that evaluates to **T** for $x \mapsto 0$, so we will pick

$$Q^{\mathcal{M}} \stackrel{\text{def}}{=} \{0\}$$

We can now state that the formula ϕ is satisfiable by presenting:

- a universe of discourse $A = \{0, 1\}$
- a model $\mathcal{M}$ with predicates $\mathcal{P} = \{P^{\mathcal{M}}, Q^{\mathcal{M}}\}$ such that $P^{\mathcal{M}} \stackrel{\text{def}}{=} \{1\}$ and $Q^{\mathcal{M}} \stackrel{\text{def}}{=} \{0\}$
- an environment $x \mapsto 0$

Next, we will find a model $\mathcal{M}'$ in which ϕ is not satisfiable. This is harder because we must find a model $\mathcal{M}'$ in which $\neg\,\exists x\,(P(x) \to Q(x))$ is satisfiable.

We can approach this problem by observing that, in predicate logic, the formula $\neg\,\exists x\,\psi$ is equivalent to the formula $\forall x\,\neg\,\psi$. We thus need to find a model $\mathcal{M}'$ in which

$$\forall x\,\neg\,(P(x) \to Q(x))$$

is satisfied.

Let us use the same universe as we used previously, so $A' = \{0, 1\}$.

Examining the formula carefully, we see that it is satisfied if the predicate $P^{\mathcal{M}'}$ always evaluates to **T** and the predicate $Q^{\mathcal{M}'}$ always evaluates to **F**; for a "fresh" free variable z, this would make

$$P^{\mathcal{M}'}(z) \to Q^{\mathcal{M}'}(z)$$

always evaluate to **F**, so its negation would always evaluate to **T**, so the universal quantification would always evaluate to **T**.

One way to think of $P^{\mathcal{M}'}$ always evaluating to **T** is that it means that $P^{\mathcal{M}'}$ is true of every member of A'. If we have the predicate $P^{\mathcal{M}'}$ *equal to* the universe A', then any argument $z \in A'$ would also be in $P^{\mathcal{M}'}$, so the predicate would always evaluate to **T**. We would propose

$$P^{\mathcal{M}} \stackrel{\text{def}}{=} \{0, 1\} \quad \text{or} \quad P^{\mathcal{M}} \stackrel{\text{def}}{=} A'$$

To have the predicate $Q^{\mathcal{M}'}$ always evaluate to **F** means it is never true, so in set notation we would propose

$$Q^{\mathcal{M}} \stackrel{\text{def}}{=} \{\}$$

Our proposed model $\mathcal{M}'$ that we have devised is:

- a universe of discourse $A' = \{0, 1\}$
- a model $\mathcal{M}'$ with predicates $\mathcal{P} = \{P^{\mathcal{M}'}, Q^{\mathcal{M}'}\}$ such that $P^{\mathcal{M}'} \stackrel{\text{def}}{=} A'$ and $Q^{\mathcal{M}'} \stackrel{\text{def}}{=} \{\}$

We can now assert that in this model $\mathcal{M}'$, for all possible environments $l(z) = a$ where $a \in A$, the semantic entailment

$$\models_{[z \mapsto a]} \neg (P(z) \to Q(z))$$

holds.

This is equivalent to showing that, in this model $\mathcal{M}'$, the semantic entailment

$$\models \neg \exists x \, (P(x) \to Q(x))$$

holds.

Our approach to showing satisfiability was to find some model $\mathcal{M}$ in which a formula ϕ evaluates to **T**. This was not difficult for existential quantification of a simple implication formula. Later in this course, we will discuss why we can never find an algorithm that determines satisfiability of an arbitrary formula in predicate logic.