Universal Quantification

Text Correspondence: pp. 109–112

Formal reasoning, or proofs, of sequents involving quantifiers begins with some simple observations. The first observation is that, if a formula is true for all values of a variable, then it is true for some specific value of the variable. We are familiar with this concept from elementary algebra, where a variable in an equation can be replaced with a numerical value; here, we extend the concept to a logical formula.

Consider a formula $\phi$ that has a variable $x$, and a term $t$ that has $t$ free for $x$ in $\phi$. If the quantified formula

$$\forall x \phi$$

is true, then the substitution of $t$ for $x$, which is

$$\phi[t/x]$$

must also be true. We must be careful here: the formula $\phi$ has $t$ free for $x$, but the formula $\forall x \phi$ has $x$ in its scope. We are saying that a universal formula can be “instantiated” with a new term, within the scope of the quantifier.

Proof Rule: Universal Elimination, $\forall x \vdash \forall x e$

$$\frac{
\forall x \phi \\
\forall x e
}{\phi[t/x]}$$

The use of $\forall x e$ can be illustrated using a simple sequent.
Example

Consider the situation where we know that a predicate $P(\cdot)$ is true of a specific value $a$; technically, $a$ is a constant function that evaluates to a value. We say that $a$ has property $P$.

Suppose further that, for any value of a variable $x$, if $x$ has property $P$ then $x$ does not have property $Q$. We naturally want to conclude that $a$ does not have property $Q$. The sequent for this example is

$$P(a), \forall x(P(x) \rightarrow \neg Q(x)) \vdash \neg Q(a) \quad (18.1)$$

The proof strategy is that we can replace the bound occurrence of $x$ in the second premise with the definite term $a$. We will annotate our proof in more detail than is recommended by the textbook, so that our proof of Argument 18.1 is

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<thead>
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<tbody>
<tr>
<td>1</td>
<td>$P(a)$</td>
<td>premise</td>
</tr>
<tr>
<td>2</td>
<td>$\forall x(P(x) \rightarrow \neg Q(x))$</td>
<td>premise</td>
</tr>
<tr>
<td>3</td>
<td>$P(a) \rightarrow \neg Q(a)$</td>
<td>$\forall x \quad e \quad 2$ \quad using $\phi[a/x]$</td>
</tr>
<tr>
<td>4</td>
<td>$\neg Q(a)$</td>
<td>$\rightarrow \quad e \quad 3, ; 1$</td>
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The next rule of predicate logic is based on the idea that, if we can prove something for a newly introduced variable, then because the proof does not depend on any specific value or binding of the new variable then the proof applies to all values of the variable.

More formally, suppose that we introduce a “fresh” variable $x_0$ that is free for $x$ in a formula $\phi$. When we have proved $\phi$ using the variable $x_0$, we have performed an extension of the propositional logic rule $\rightarrow i$. The logic, in a proof, would be

$$\begin{array}{c}
h_0 \\
\vdots \\
\phi[x_0/x]
\end{array}$$

where the last line is how we state that the “fresh” variable $x_0$ can be substituted. From this, we want to be able to conclude that $\phi$ can be universally quantified.

Before defining the new proof rule, we can consider an example of how we might want to use such a rule.
**Example**

From propositional logic, we know that \( p \lor \neg p \) is a tautology. A straightforward extension into predicate logic would be, for a variable \( x \) and a property \( Q \), that for all \( x \) either \( x \) has the property \( Q \) or it does not; in symbols this is \( \forall x (Q(x) \lor \neg Q(x)) \).

Our general way of reasoning is that, if we use the tautology so that it has a “new” free variable \( x_0 \) in it, then we can universally quantify the tautology. We want to be able to introduce the new variable and draw our conclusion from it. The argument we want to use has no premises; the sequent has only a conclusion

\[
\vdash \forall x (Q(x) \lor \neg Q(x))
\]

Our specific line of reasoning is this: from the Law of Excluded Middle, with a new variable \( x_0 \), we can assert \( Q(x_0) \lor \neg Q(x_0) \). Because we have not specified a specific value for \( x_0 \), we want to be able to conclude that the disjunction is true for all values of \( x_0 \). The form of the argument we want to use is

\[
\begin{array}{c}
x_0 \quad Q(x_0) \lor \neg Q(x_0) \quad \text{LEM} \\
\hline
\forall x Q(x) \lor \neg Q(x)
\end{array}
\]

To be able to use this kind of argument, we need a new proof rule.

**Proof Rule: Universal Introduction, \( \forall x \ i \)**

\[
\begin{array}{c}
x_0 \\
\vdots \\
\phi[x_0/x] \\
\hline
\forall x \phi \quad \forall x \ i
\end{array}
\]

One way to express this rule in English is: introducing a new variable \( x_0 \), if we can prove a formula \( \phi \) that has the new variable \( x_0 \) substituted into it, then we can conclude that \( \phi \) is valid when universally quantified.
**Example**

Consider an example where we assert that every \(x\) that has property \(P\) also has property \(Q\). We must be careful in translating this English into symbols, so that we get

\[
\forall x (P(x) \rightarrow Q(x))
\]  
(18.2)

We may further assert that every \(x\) has property \(P\). From this, we want to conclude that every \(x\) has property \(Q\). Combining Argument 18.2 with translations of the second assertion and the conclusion, we want to prove the sequent

\[
\forall x (P(x) \rightarrow Q(x)), \forall x P(x) \vdash \forall x Q(x)
\]  
(18.3)

After writing the premises, we will assume the use of a new variable \(z\). Using universal elimination, the premise of Argument 18.2 allows us to deduce that \(P(z) \rightarrow Q(z)\). Using universal elimination again, the second premise \(\forall x P(x)\) allows us to deduce \(P(z)\). Applying the rule Modus Ponens, which is implication elimination, we can deduce \(Q(z)\).

Because the variable \(z\) is new and is also free, we can use universal introduction to deduce that \(\forall x Q(x)\). Our proof is:

\[
\begin{align*}
1 & \quad \forall x (P(x) \rightarrow Q(x)) \quad \text{premise} \\
2 & \quad \forall x P(x) \quad \text{premise} \\
3 & \quad z \quad \text{assumption} \\
4 & \quad P(z) \rightarrow Q(z) \quad \forall x \ e \ 1 \\
5 & \quad P(z) \quad \forall x \ e \ 2 \\
6 & \quad Q(z) \quad \rightarrow \ e \ 4, 5 \\
7 & \quad \forall x Q(x) \quad \forall x \ i \ 3-6 \ \text{where} \ \phi \ \text{is} \ Q(x)
\end{align*}
\]

In English, one way to read \(\phi[t/x]\) is: \(\phi\) in which every free occurrence of \(x\) is replaced by \(t\).

In this example, we introduced a variable \(z\) that led to the formula of Line 6; because this \(z\) does not have a specified value, the formula \(\phi\) is true for all values, so we can universally quantify the formula \(\phi\).

**Exercise**

Students should now be able to modify the above example to prove the validity of the sequent

\[
\forall x (P(x) \rightarrow Q(x)) \vdash \forall x P(x) \rightarrow \forall x Q(x)
\]  
(18.4)

Hint: consider the propositional logic rule \(\rightarrow i\).