School of Computing  
CISC/CMPE 204  
Logic In Computer Science  

Test # 2, Paper A – ANSWERS  

October 18, 2016  

Please answer only in the answer boxes provided. You may use the back of the pages as scrap paper.  
This is a closed-book test. No computers or calculators are allowed.  
A reference page is provided at the end of the test. You may use only these rules of inference.  
Should a question be unclear or ambiguous, you should make a reasonable interpretation and state what you have assumed.  
To be eligible for re-marking, this tests must be answered entirely in indelible (unerasable) ink. If erasable ink or pencil is used, then the test will be marked exactly once. 

Do not begin until instructed to do so.

For Marker Use Only

<table>
<thead>
<tr>
<th>Question</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Question 1</td>
<td>15</td>
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<tr>
<td>Question 2</td>
<td>10</td>
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<tr>
<td>Question 3</td>
<td>10</td>
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<td>Question 4</td>
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<td>Total</td>
<td>40</td>
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Question 1 - This question is a test of understanding the semantics of a formula in propositional logic. For the formula $\phi$ that is

$$
\left( (\neg q \land p) \lor (q \land \neg p) \right) \land r
$$

Justify each step of your proof.

(a) Construct the truth table for $\phi$

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$r$</th>
<th>$\neg p$</th>
<th>$\neg q$</th>
<th>$q \land \neg p$</th>
<th>$(\neg q \land p) \lor (q \land \neg p)$</th>
<th>$(\neg q \land p) \lor (q \land \neg p) \land r$</th>
</tr>
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<tbody>
<tr>
<td>T</td>
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</tr>
</tbody>
</table>

6 points total for this part:
- 3 for correct final column
- 2 for correct intermediate columns (all included)
- 1 for setting up proper truth table with 3 variables

(b) Construct the conjunctive Normal Form (CNF) for $\phi$

You may use either the truth table or the conversion algorithm. Show all of your steps.

Truth Table (for all rows where F, take the disjunction of the negation of each value, then take the conjunction of these disjunctions):

$$
\left( (\neg p \lor \neg q \lor \neg r) \land (\neg p \lor q \lor r) \land (p \lor \neg q \lor r) \land (p \lor q \lor \neg r) \right)
\land (p \lor q \lor r)
$$

Algorithm:
1. Remove Implications
2. Cancel Double Negations
3. Distribute All Negations:
4. Distribute all Disjunctions of Conjunctions to Conjunction of Disjunctions:

Since this can be done in numerous orders, you will need to check their logic if this approach was taken, remembering the applications of DeMorgan’s Law:

$$
((\phi \land \psi) \lor \eta) \equiv ((\phi \lor \eta) \land (\psi \lor \eta))
$$

$$
((\phi \lor \psi) \land \eta) \equiv ((\phi \land \eta) \lor (\psi \land \eta))
$$

9 points total for this part, regardless of approach taken: 4 marks for correct final answer, 3 marks for demonstrating understanding of approach (by showing or explicitly stating), 2 marks for following through with approach but not getting fully correct answer – for possible marks being 9 for perfect, 5 for setting up and starting to solve but not finishing correctly, 3 for providing the algorithm or steps involved.

15 points
**Question 2** - This question is a test of translating natural language into the language of predicate logic. For this question, use these predicate symbols, function symbols, and constants:

- $D(x)$: $x$ is a student
- $E(x)$: $x$ is an exam
- $P(x, y)$: $x$ is a problem on exam $y$
- $S(x, y)$: $x$ solves $y$
- $f(x, y)$: the $x^{th}$ problem on exam $y$
- $h(x)$: the hardest problem on exam $x$

1: one
2: two
3: three

Translate each of the following sentences into predicate logic.

(a) No student solves problem three on exam one

$$
\neg \exists x (D(x) \land S(x, f(3,1)))
$$

or

$$
\forall x (D(x) \rightarrow S(x, f(3,1)))
$$

(b) Every student who solves problem one on exam two also solves the hardest problem on exam two

$$
\forall x ((D(x) \land S(x, f(1,2))) \rightarrow S(x, h(2)))
$$

(c) If some student solves every problem on every exam, then some student solves the hardest problem on every exam

$$
\exists x (D(x) \land \forall y \forall z (S(x, f(y,z))) \rightarrow \exists x (D(x) \land \forall z (S(x, h(z))))
$$

(d) No student solves every problem on every exam

$$
\neg \exists x (D(x) \land \forall y \forall z (S(x, f(y,z))))
$$

(e) If some student solves no problems on exam one, then some student will not solve the hardest problem on exam one

$$
\exists x (D(x) \land \forall y (\neg S(x, f(y,1))) \rightarrow \exists x (D(x) \land \neg S(x, h(1)))
$$

2 points per conversion to predicate logic – full marks for a logical statement, 1 for close, and 0.5 for a reasonable attempt. 0 marks for something that makes no sense at all.

10 points
Question 3 - This question is a test of understanding the scope of variables in predicate logic. Consider the formula

$$\exists x (P(x, y) \lor Q(y, z) \rightarrow \forall z R(z, x))$$

(a) Draw the parse tree for $\phi$

5 points for the correct parse tree as follows:

1 point for Existential quantifier being the root
1 point for the implication immediately following it
1 point for the universal quantifier and the disjunction as its children
1 point for having $P, Q, R$ in correct place
1 point for having the 2 variables as children of the correct predicates

(b) Identify every free and bound variable in $\phi$

See above.

3 Points total – 0.5 for each correctly identified variable.

(c) Complete the substitution $\phi[g(a, z)/y]$

Since $y$ is free in both occurrences, and neither $a$ nor $z$ become bound, the substitution is:

$$\exists x (P(x, g(a, z)) \lor Q(g(a, z), z) \rightarrow \forall z R(z, x))$$

2 points for correctly performing the substitution.

10 points
Question 4 – This question deals with the concept of Horn Satisfiability. Is the following Horn Clause satisfiable?

\[ \phi = (\top \rightarrow x) \land (\top \rightarrow s) \land (z \rightarrow \bot) \land (w \land x \land q \rightarrow z) \land (w \rightarrow q) \land (T \rightarrow v) \land (v \rightarrow w) \]

Show your work step by step and justify your conclusion.

Answer

Mark all \( T \):

\[ (\top \rightarrow x) \land (\top \rightarrow s) \land (z \rightarrow \bot) \land (w \land x \land q \rightarrow z) \land (w \rightarrow q) \land (T \rightarrow v) \land (v \rightarrow w) \]

Mark all consequents (and all occurrences of them) where the antecedent is marked (repeat):

\[ (\top \rightarrow x) \land (\top \rightarrow s) \land (z \rightarrow \bot) \land (w \land x \land q \rightarrow z) \land (w \rightarrow q) \land (T \rightarrow v) \land (v \rightarrow w) \]

- \( x \), \( s \), and \( v \) marked

\[ (\top \rightarrow x) \land (\top \rightarrow s) \land (z \rightarrow \bot) \land (w \land x \land q \rightarrow z) \land (w \rightarrow q) \land (T \rightarrow v) \land (v \rightarrow w) \]

- \( w \) marked

\[ (\top \rightarrow x) \land (\top \rightarrow s) \land (z \rightarrow \bot) \land (w \land x \land q \rightarrow z) \land (w \rightarrow q) \land (T \rightarrow v) \land (v \rightarrow w) \]

- \( q \) marked

\[ (\top \rightarrow x) \land (\top \rightarrow s) \land (z \rightarrow \bot) \land (w \land x \land q \rightarrow z) \land (w \rightarrow q) \land (T \rightarrow v) \land (v \rightarrow w) \]

- \( z \) marked

\[ (\top \rightarrow x) \land (\top \rightarrow s) \land (z \rightarrow \bot) \land (w \land x \land q \rightarrow z) \land (w \rightarrow q) \land (T \rightarrow v) \land (v \rightarrow w) \]

- \( \bot \) marked

\( \bot \) has been marked, therefore \( \phi \) is unsatisfiable.

5 points:

- 2 points for correct conclusion of unsatisfiable
- 1 point for justification that it is unsatisfiable because bottom is marked
- 2 points for having (at least) these 6 steps, they may have done each line individually, this is OK as well
Reference: Rules of Deduction

The basic rules in propositional logic:

<table>
<thead>
<tr>
<th>introduction</th>
<th>elimination</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\land$</td>
<td></td>
</tr>
<tr>
<td>$\phi \land \psi$</td>
<td>$\phi \land \psi$</td>
</tr>
<tr>
<td>$\phi \land \psi$</td>
<td>$\phi \land \psi$</td>
</tr>
<tr>
<td>$\phi \lor \psi$</td>
<td>$\phi \lor \psi$</td>
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<td>$\phi \lor \psi$</td>
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<td>$\phi \rightarrow \psi$</td>
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<td>$\phi \rightarrow \psi$</td>
</tr>
<tr>
<td>$\bot$</td>
<td>$\bot$</td>
</tr>
<tr>
<td>$\phi$</td>
<td>$\phi$</td>
</tr>
<tr>
<td>$\neg i$</td>
<td>$\neg e$</td>
</tr>
</tbody>
</table>

Some derived rules in propositional logic:

- **MT**: $\phi \rightarrow \psi$ $\neg \psi$ $\neg \phi$
- **PBC**: $\neg \phi$ $\bot$
- **LEM**: $\phi \lor \neg \phi$
- **i**: $t_1 = t_2$ $\phi[t_1/x]$
- **e**: $\phi[t_2/x]$