School of Computing
CISC/CMPE 204
Logic In Computer Science

Test # 4

November 15, 2016

Please answer only in the answer boxes provided. You may use the back of the pages as scrap paper. This is a closed-book test. No computers or calculators are allowed.

A reference page is provided at the end of the test. You may use only these rules of inference.

Should a question be unclear or ambiguous, you should make a reasonable interpretation and state what you have assumed.

To be eligible for re-marking, this tests must be answered entirely in indelible (unerasable) ink. If erasable ink or pencil is used, then the test will be marked exactly once.

Do not begin until instructed to do so.

| Question 1 | 15 |
| Question 2 | 10 |
| Question 3 | 10 |
| Question 4 | 5  |
| **Total**  | **40** |
Question 1 - This question is a test of understanding predicate models and formula satisfaction. Consider the two formulas $\phi_1$ and $\phi_2$ that are:

$$\phi_1 = \forall x \forall y \left( P(x, x) \lor Q(x, y) \right)$$

$$\phi_2 = \forall x \left( P(x, x) \rightarrow Q(x, x) \right)$$

Note: $P$ and $Q$ are predicates.

(a) Find a model $\mathcal{M}$ that satisfies $\phi_1$ and does not satisfy $\phi_2$

There are numerous answers for this question that will hold for the first formula and not the other. The most common of which is $P$ being equality and $Q$ being inequality. However you will need to reason out for other examples. They, at minimum, must provide a universe of discourse, and definitions for $P$ and $Q$:

$A = \{0,1\}$

$P(x,y) = (x = y) = \{(0,0),(1,1)\}$

$Q(x,y) = \neg(x = y) = \{(0,1),(1,0)\}$

Alternatively, another popular solution is:

$A = \{0\}$

$P(x,y) = \{(0,0)\}$

$Q(x,y) = \{\}$

7.5 Marks Total:
- 0.5 for defining $A$
- 1 mark for defining any $P$
- 1 mark for defining any $Q$
- 3 marks if it satisfies formula 1
- 2 marks if it does not satisfy formula 2

(b) Find a model $\mathcal{M}'$ that satisfies $\phi_2$ and does not satisfy $\phi_1$

Again there will be numerous models that work for this case, but the most popular reasoning is to get $P$ to be false when the values are the same, which makes the implication in formula 2 always evaluate to true, and then you only need to worry about a definition of $Q$ that never evaluated to true, so that formula 1 is not satisfiable – the simplest of which is the empty set for $Q$:

$A = \{0,1\}$

$P = \neg(x = y) = \{(0,1),(1,0)\}$

$Q = \{\}$

7.5 Marks Total:
- 0.5 for defining $A$
- 1 mark for defining any $P$
- 1 mark for defining any $Q$
- 3 marks if it satisfies formula 2
- 2 marks if it does not satisfy formula 1

15 points
Question 2 - This question is a test of understanding that a set of predicate formulas is consistent. This set of formulas is not consistent:

1. \( \forall x (P(x) \rightarrow \forall y \neg Q(y)) \)
2. \( \exists x P(x) \)
3. \( \forall x Q(x) \)

Either show that there is no model that holds for all of these formulas simultaneously, or show that the formulas lead to a contradiction.

Note: \( P \) and \( Q \) are predicates.

Answer

As usual, there will be slight variations in ordering, but the solution should look similar to this proof.

- 1 point for setting it up as a proof with bottom as conclusion
- 3 points for setting up existential elimination, with \( P(x) \) instantiated, and bottom as conclusion, in a box
- 2 points for universal eliminations (1 point each for each universal premise)
- 2 points implication elimination
- 1 point for universal elimination of the universal quantifier on line 5
- 1 point for using negation elimination to arrive at bottom

If they attempted to “show that there is not model that holds for all of these formulas simultaneously” then there are no marks that can be awarded, since to show this they would need to show every model (an infinite number) does not hold for at least one. It was explicitly covered in class that this type of question can be done via proof.
Question 3 - This question is a test of examining assertions about a model in predicate logic, and determining if the assertion holds. Consider the following model of base four arithmetic from class:
(recall P refers to even, Q refers to odd, i returns the initial value, and f returns the next integer, mod 4)

For each of the following assertions within the model:
(a) state whether the assertion holds or does not hold – justify your answer with reasoning – use examples or counter examples to assist you
(b) translate the symbolic logic to English

Answer
(a) \( \forall x \ (P(x) \lor Q(x)) \)
   x is 0, then P(0) is true, thus the whole formula is satisfiable
   x is 1, then Q(1) is true, thus the whole formula is satisfiable
   x is 2, then P(2) is true, thus the whole formula is satisfiable
   x is 3, then Q(3) is true, thus the whole formula is satisfiable
   since this is satisfiable for all x, assertion (a) holds

(b) \( \exists x \ (P(x) \land Q(x)) \)
   x is 0, then Q(0) is false, thus the whole formula is not satisfiable
   x is 1, then P(1) is false, thus the whole formula is not satisfiable
   x is 2, then Q(2) is false, thus the whole formula is not satisfiable
   x is 3, then P(3) is false, thus the whole formula is not satisfiable
   since this is not satisfiable for any x, assertion (b) does not hold

(c) \( \exists x \ (P(x) \rightarrow Q(f(x))) \)
Try letting x be 0
P(0) evaluates to T, f(0) returns 1, and Q(1) evaluates to T
The equation is an implication of T \( \rightarrow \) T, which holds
This is one positive example, showing there is at least one x

Therefore, assertion (c) holds

(d) \( \forall x \ (Q(x) \rightarrow Q(f(f(x)))) \)

<table>
<thead>
<tr>
<th>X</th>
<th>Q(x)</th>
<th>f(x)</th>
<th>f(f(x))</th>
<th>Q(f(f(x)))</th>
<th>( \rightarrow )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>F</td>
<td>1</td>
<td>2</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>1</td>
<td>T</td>
<td>2</td>
<td>3</td>
<td>T</td>
<td>T</td>
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<tr>
<td>3</td>
<td>T</td>
<td>0</td>
<td>1</td>
<td>T</td>
<td>T</td>
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</tbody>
</table>

Holds for all x, thus assertion (d) holds
Translation
For all numbers, if a number is odd, the number 2 after it (mod 4) will be odd

(e) \( \forall x \ (P(x) \rightarrow Q(x)) \)
Try letting x be 0
P(0) evaluates to T, Q(0) evaluates to F
The equation is an implication of T \( \rightarrow \) F which does not hold
This is one counter example in a universal quantification

Therefore, assertion (e) does not hold

Translation
For all numbers, if a number is even, that number is odd
Marks for Q3:

2 marks each part:
- 0.5 marks for the correct conclusion (hold/not hold)
- 0.5 marks for the correct (or very close) translation
- 1 mark for justifying their answer, which must be of the following form:
  o If claim is universal quantifier holds, they have to show for all values that it holds
  o If claim is universal quantifier does not hold, one counter example provided
  o If claim is existential quantifier holds, one positive example provided
  o If claim is existential quantifier does not hold, then all valuations must not hold
**Question 4** - This question is a test of understanding the equality predicate and semantic models.

Consider the sequent:

$$\exists x (P(x) \land Q(x)) \vdash \exists y \forall z ((x = y) \rightarrow (P(y) \land Q(y)))$$

Either prove that the sequent holds, using the rules of predicate logic, or provide a model $\mathcal{M}$ in which the premise evaluates to $T$ and the conclusion evaluates to $F$.

Note: $P$ and $Q$ are predicates.

### Answer

<table>
<thead>
<tr>
<th>Step</th>
<th>Formula</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\exists x (P(x) \land Q(x))$</td>
<td>premise</td>
</tr>
<tr>
<td>2</td>
<td>$w \quad (P(w) \land Q(w))$</td>
<td>assume</td>
</tr>
<tr>
<td>3</td>
<td>$z \quad (P(z) \land Q(z))$</td>
<td>assume</td>
</tr>
<tr>
<td>4</td>
<td>$(w = z)$</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$(P(z) \land Q(z))$</td>
<td>= elim</td>
</tr>
<tr>
<td>6</td>
<td>$(w = z) \rightarrow (P(z) \land Q(z)))$</td>
<td>$\rightarrow$ intro 4-5</td>
</tr>
<tr>
<td>7</td>
<td>$\forall y ((w = y) \rightarrow (P(y) \land Q(y)))$</td>
<td>Univ intro 3-6</td>
</tr>
<tr>
<td>8</td>
<td>$\exists x \forall y ((x = y) \rightarrow (P(y) \land Q(y)))$</td>
<td>Exist intro 7</td>
</tr>
<tr>
<td>9</td>
<td>$\exists x \forall y ((x = y) \rightarrow (P(y) \land Q(y)))$</td>
<td>Exist elim 1, 2-8</td>
</tr>
</tbody>
</table>

- 1 point for correctly using existential elim to instantiate premise and arrive at same item as conclusion
- 1 point for having existential intro be second last step
- 1 point for properly using universal intro to get to line 7
- 1 point of using implication introduction to get line 6
- 1 point for equality substitution to get from 2 to 5, using equality on line 4

Beyond this answer there may be other proofs – use your best judgment to assign a mark out of 5 accordingly.

5 points