The proof rules of predicate logic include the rules of propositional logic. Syntactically, all of the previous rules apply; we are using them on different formulas so the semantics will be more general.

The first rules we need are for the equality of terms. Substitution of variables was not available in propositional logic and we need to ensure that substitution gives us three properties when term $t_1$ is equal to term $t_2$:

**Reflexive:** A term is equal to itself, so $t = t$

**Symmetric:** Equality is independent of order, so $(t_1 = t_2) \vdash (t_2 = t_1)$

**Transitive:** Equality extends to other terms, so $(t_1 = t_2), (t_2 = t_3) \vdash (t_1 = t_3)$

These properties, and others, are ensured by the two proof rules for equality.

**Proof Rule: Equality Introduction, =i**

\[
\begin{array}{c}
t = t
\end{array}
\]

This rule is an axiom because it does not require any premises. It encodes reflexivity, by stating that it is always true that a term is equal to itself. With one additional rule, equality introduction can be used to prove that equality has the three desired properties.

The additional equality rule is simply stated. If two terms are equal, and the first is substituted for a variable in a formula, then it is valid to conclude that the second term can also be substituted in the same formula.

**Proof Rule: Equality Elimination, =e**

\[
\begin{array}{c}
t_1 = t_2 \\
\phi[t_1/x] \\
\phi[t_2/x]
\end{array} \quad \begin{array}{c}
Provided that t_1 is free for x in \phi \\
that t_2 is free for x in \phi
\end{array}
\]

It is very important to observe that both substitutions must be possible, that is, we must have $t_1$ free for $x$ in $\phi$ and $t_2$ free for $x$ in $\phi$. These are requirements for substitution, and can be thought of as side conditions for the proof rule of equality introduction.
The properties of symmetry and transitivity are simple to prove. The textbook has short proofs for these properties – just 3 lines each – that students are encouraged to understand thoroughly. We can examine strategies for these proofs.

Example
Consider the symmetry property as a sequent:

\[ t_1 = t_2 \vdash t_2 = t_1 \]

The rule of \( =i \) does not look too promising so we can turn to the application of \( =e \). This requires two premises, an equality and a formula that is either true or is a premise; here, we have only one premise in the original sequent, so we need a tautology for the second premise of \( =e \).

We also see that we need to arrive at the conclusion, \( t_2 = t_1 \). We can do this using \( =e \) if we have a formula in a variable \( x \) such that, if we substitute \( t_1 \) for \( x \) we get a tautology, and if we substitute \( t_2 \) for \( x \) we get the conclusion of the original sequent.

One formula that works is where \( \phi \) is \( x = t_1 \). Substituting in \( t_1 \), we get

\[ \phi[t_1/x] \text{ is } t_1 = t_1 \]

and substituting in \( t_2 \), we get

\[ \phi[t_2/x] \text{ is } t_2 = t_1 \]

We can now construct the proof. We will go further than the textbook and annotate the use of the \( =e \) rule with the formula that we use. This gives us the proof

1. \( t_1 = t_2 \)  premise
2. \( t_1 = t_1 \)  \( =i \)
3. \( t_2 = t_1 \)  \( =e \) 1, 2  where \( \phi \) is \( x = t_1 \)
Example
Consider the transitivity property as a sequent. To reduce the potential for confusion in the names of variables in the proof and in the rules of logic, we will write this slightly differently than the textbook. Our sequent is

\[ s_1 = s_2, s_2 = s_3 \vdash s_1 = s_3 \]

We will do in predicate logic what we did in propositional logic, which is begin by assuming the premises. A useful strategy for a proof is to write down the conclusion of the sequent, which is \( s_1 = s_3 \). To apply the rule \( =e \), we need to arrive at the conclusion using only the rules we know; we will also try to avoid using the property of symmetry, so that we can show how transitivity is a property independent of symmetry.

We need to construct a formula \( \phi \) with two properties. The first property is that, when we substitute in one term, we get either a premise or a tautology. The second property is that, when we substitute in a different term, we get the conclusion. The conclusion formula is \( s_1 = s_3 \) so there are two immediate candidates for \( \phi \): we could try \( s_1 = x \) or \( x = s_3 \). We will try the first formula.

What term would substitute for \( x \) in \( \phi \) to make \( s_1 = x \) either a premise or a tautology? The term \( s_2 \) would work, because \( s_1 = s_2 \) is a premise; the substitution is

\[ \phi[s_2/x] \text{ is } s_1 = s_2 \]

What term would substitute for \( x \) in \( \phi \) to make \( s_1 = x \) the conclusion of the argument? The term \( s_3 \) would work, because its substitution is

\[ \phi[s_3/x] \text{ is } s_1 = s_3 \]

To use the exact order of the rule \( =e \), we will construct our proof so that the premises are laid out with this rule in mind. Our proof is

1. \( s_2 = s_3 \) premise
2. \( s_1 = s_2 \) premise
3. \( s_1 = s_3 =e 1, 2 \) where \( \phi \) is \( s_1 = x \)
Example

Consider trying to prove the sequent
\[ t_1 = t_2 \vdash (t + t_1) = (t + t_2) \]

The premise of this sequent looks like one of the two premises of \(=\text{e}\). As above, we need to construct a formula \(\phi\) with two properties. The first property is that, when we substitute in one term, we get either a premise or a tautology. The second property is that, when we substitute in a different term, we get the conclusion.

There is only one premise to the sequent, so we need a formula that (a) substitutes in one term to produce a tautology and (b) substitutes in the other term to produce the conclusion.

The form of the \(=\text{e}\) rule suggests that we end up substituting \(t_2\) for \(x\). A simple candidate formula that has this property is
\[ \phi \text{ is } t + t_1 = t + x \]  
(15.1)

When we substitute \(t_2\) for \(x\) in Formula 15.1, we get
\[ \phi[t_2/x] \text{ is } t + t_1 = t + t_2 \]
which is the conclusion. When we substitute \(t_1\) for \(x\) in Formula 15.1, we get
\[ \phi[t_1/x] \text{ is } t + t_1 = t + t_1 \]

This is a tautology involving equality, which can be concluded with a single use of the rule \(=\text{i}\). With careful annotation, our proof is

\begin{align*}
1 & \quad t_1 = t_2 \quad \text{premise} \\
2 & \quad (t + t_1) = (t + t_1) \quad =\text{i} \quad \text{using the term } (t + t_1) \\
3 & \quad t + t_1 = t + t_2 \quad =\text{e 1, 2} \quad \text{where } \phi \text{ is } t + t_1 = t + x
\end{align*}

This kind of reasoning is likely to differ greatly from the kind of reasoning that a student has used so far in mathematics. Whereas we might naturally reason:

\begin{align*}
1 & \quad t_1 = t_2 \quad \text{premise} \\
2 & \quad (t + t_1) = (t + t_1) \quad \text{tautology} \\
3 & \quad t + t_1 = t + t_2 \quad \text{substitute } t_2 \text{ into the second occurrence of } t_1
\end{align*}

the rules of predicate logic, so far, have not justified this line of reasoning. It is remarkable that, from the simple rules of equality, we can deduce fairly powerful results.

The sequent that we have proved is a specific form of a more general theorem, which is that if two terms are equal then a property of the first term is also a property of the second term. The general form is
\[ t_1 = t_2 \vdash P(t_1) = P(t_2) \]

What formula \(\phi\) could be used to prove this more general result? (Finding such a formula is left as an exercise for the student.)