In natural deduction, we have used “assumption boxes” 4 ways in propositional logic and 2 ways in predicate logic. Let us review these uses so that we are sure that we understand them.

**Implication Introduction**

This is used when we want to deduce a formula $\phi \rightarrow \psi$. We assume $\phi$ and, using deduction rules, assert $\psi$; from this we can conclude that $\phi$ implies $\psi$.

**Example:** “Suppose that the falsity of $q$ implies the falsity of $p$; we can conclude that $p$ implies $q$.”

\[
\neg q \rightarrow \neg p \vdash p \rightarrow q \quad H&R, p. 13
\]

**Proof Strategy:** We can prove this using double negation and MT, Modus Tollens. To begin, we write the premise and the conclusion; we leave space for the intermediate lines of the proof.

1  $\neg q \rightarrow \neg p$  premise

\[
\vdash p \rightarrow q
\]

The conclusion is an implication so we “open” an $\rightarrow i$ box. **Important:** we put the antecedent at the top and the consequent at the bottom so that we know exactly what we are assuming and what we wish to deduce.

1  $\neg q \rightarrow \neg p$  premise

2  \vspace{5mm}

2  $p$  assumption

\[
\vdash q
\]

6  $p \rightarrow q$  $\rightarrow i$ 2–?

We can now fill in the rest of the proof. From $p$ we can deduce $\neg \neg p$; applying the rule MT we can deduce $\neg \neg q$; and from this we can deduce $q$. The complete proof looks like
Key Concept: The $\rightarrow i$ rule uses “backward” logical reasoning. When we open the $\rightarrow i$ “box”, we put the antecedent at the top – as an assumption – and the consequent at the bottom – as the desired conclusion. This constrains our reasoning so that the $\rightarrow i$ rule is applied correctly.

Disjunction Elimination
This is used when we know a disjunctive formula $\phi \lor \psi$ and we want to deduce a formula $\chi$. The idea is: if we assume $\phi$ and can deduce $\chi$, and if we assume $\psi$ and we can deduce $\chi$, then – because either $\phi$ or $\psi$ is true and even though we do not know which one is true – we can conclude that $\chi$ is true.

Example:
“Suppose that $p$ implies $q$. We can conclude that either $p$ is false or $q$ is true.”

\[
p \rightarrow q \vdash \neg p \lor q \quad H&R, p. 26
\]

Proof Strategy: We can reason this using the Law of Excluded Middle. If $p$ is true then we can use $\rightarrow e$ to deduce $q$, from which we can deduce $\neg p \lor q$; if $p$ is false then we can directly deduce $\neg p \lor q$. Our proof begins by writing the premise and the conclusion.

\[
1 \quad p \rightarrow q \quad \text{premise} \\
\quad \vdash \\
\quad \neg p \lor q
\]

The textbook proves this sequent by using LEM to introduce the proposition $p \lor \neg p$, which we can always write down because LEM does not refer to any previous lines of a proof.

\[
1 \quad p \rightarrow q \quad \text{premise} \\
2 \quad p \lor \neg p \quad \text{LEM} \\
\quad \vdash \\
\quad \neg p \lor q \quad \lor e 2, 3–5, 6–7
\]
Next, to apply the rule $\lor$e, we open 2 assumption boxes: one for the left disjunct $p$ and one for the right disjunct $\neg p$. We must put the same formula at the bottom of each box, according to the rule. For now, we will not number the lines within the assumption boxes. We do, however, know that the conclusion of the proof is arrived at using the $\lor$e rule so we can note that in the “rules” column.

\begin{align*}
1 & \quad p \rightarrow q \quad \text{premise} \\
2 & \quad p \lor \neg p \quad \text{LEM} \\
\begin{array}{ll}
& p \quad \text{assumption} \\
& \vdots \\
& \neg p \lor q
\end{array}
\end{align*}

\begin{align*}
\neg p \quad \text{assumption} \\
\neg p \lor q \\
\neg p \lor q \quad \lor e 2, \neg?, \neg?
\end{align*}

The proof is nearly complete. In the left-disjunct assumption box, we apply $\rightarrow$e and then apply $\lor i_2$. In the right-disjunct assumption box, we apply $\lor i_1$.

\begin{align*}
1 & \quad p \rightarrow q \quad \text{premise} \\
2 & \quad p \lor \neg p \quad \text{LEM} \\
3 & \quad p \quad \text{assumption} \\
4 & \quad q \quad \rightarrow e 1, 3 \\
5 & \quad \neg p \lor q \quad \lor i_2 4 \\
\begin{array}{ll}
6 & \neg p \quad \text{assumption} \\
7 & \neg p \lor q \quad \lor i_1 6 \\
8 & \neg p \lor q \quad \lor e 2, 3–5, 6–7
\end{array}
\end{align*}

**Key Concept:** The $\lor$e rule uses “forward” logical reasoning. When we open the two $\lor$e “boxes”, we put a disjunct at the top of each — as an assumption — and the *the same* consequent at the bottom — as the desired conclusion. This constrains our reasoning so that the $\lor$e rule is applied correctly.
**Negation Introduction and Proof By Contradiction**

These rules are quite similar; they differ in whether the conclusion is or is not a negation formula. We will go through negation introduction, \( \neg i \), from which proof by contradiction can be derived.

The rule of negation introduction \( \neg i \) says that, to prove a formula that is a negation \( \vdash \neg \phi \), we assume the “positive” formula \( \phi \). If we can deduce a contradiction \( \bot \) then our assumption \( \phi \) must be false, so \( \neg \phi \) must be true.

**Example:** “Suppose that: \( p \) implies \( q \) is true, and that \( p \) implies \( q \) is false. We can conclude that \( p \) is false.”

\[
\begin{align*}
p \to q, & p \to \neg q \vdash \neg p \\
\text{H&R, p. 22}
\end{align*}
\]

**Proof Strategy:** We can reason that, if the opposite of the conclusion is assumed, then we can deduce both \( q \) and \( \neg q \) which are contradictory.

We write the premises and conclusion, leaving space for the rest of the proof.

1. \( p \to q \) premise
2. \( p \to \neg q \) premise

Our strategy is to assume the opposite of our desired conclusion. We do this by opening an assumption box. The simplest “opposite” of \( \neg p \) is \( p \); to correctly apply the \( \neg i \) rule, we need to end with a contradiction \( \bot \). We know two additional things: the \( \neg i \) rule is used to deduce the conclusion and the \( \neg e \) rule will be used to deduce the contradiction \( \bot \). We note these in the “rules” column; the question marks are in these notes but in a proof we would temporarily leave them blank.

1. \( p \to q \) premise
2. \( p \to \neg q \) premise
3. \( p \) assumption

\[
\begin{align*}
\bot & \neg e \ ?;\
\end{align*}
\]

? \( \neg p \) \( \neg i ?;? \)
We can now finish the proof. From the assumption \( p \) we can deduce \( q \), and from \( p \) we can also deduce \( \neg q \). From these we can deduce that a contradiction is present and we have solved the problem.

\[
\begin{array}{ll}
1 & p \to q \quad \text{premise} \\
2 & p \to \neg q \quad \text{premise} \\
3 & p \quad \text{assumption} \\
4 & q \to \bot \quad \text{e} 1, 3 \\
5 & \neg q \to \bot \quad \text{e} 2, 3 \\
6 & \bot \quad \text{e} 4, 5 \\
7 & \neg p \quad \text{i} 3-6
\end{array}
\]

**Key Concept**: The \( \neg \text{i} \) rule uses “backward” logical reasoning. When we open the \( \neg \text{i} \) “box”, we put the “positive” of the conclusion at the top of the box – as an assumption – and the contradiction symbol \( \bot \) at the bottom – as the desired conclusion. This constrains our reasoning so that the \( \neg \text{i} \) rule is applied correctly.

**Universal Introduction**

This rule means that, if we can prove that a formula is true using a “fresh” variable, then it is true for all values of a variable. In practice, what we do is that we inspect the preceding lines of proof and find a variable name that is free for every line; this is the “fresh” variable. We open a box and “assume” this fresh variable, so that the variable is available only within the scope of the assumption box. We then substitute the variable into the desired conclusion and put this at the bottom of the assumption box. We can then reason, within the assumption box, using the “fresh” variable.

**Example**: “Suppose that everything has both properties \( P \) and \( Q \). We can conclude that everything has property \( Q \).

\[
\forall x (P(x) \land Q(x)) \vdash \forall x Q(x)
\]

The premises and the conclusion are straightforward.

\[
\begin{array}{l}
1 \quad \forall x (P(x) \land Q(x)) \quad \text{premise} \\
\vdots \\
\forall x Q(x)
\end{array}
\]

The conclusion is a universal quantifier, so we need to use the rule \( \forall x \text{i} \) to deduce the conclusion. We inspect the premise and the conclusion, observing that the variable \( z \) is free in
both lines; therefore $z$ is free for $x$ in both lines and it is “fresh”. We open an assumption box. We write the variable $z$ at the top. At the bottom, we write substitution of $z$ for $x$ in the conclusion. We know we will use the $\forall x$ i rule so we note that in the “rules” column. To be careful, we also note the formula we are using (the text is not strict and this is for us to learn how to use the assumption box correctly).

$$
\begin{array}{c}
1 & \forall x (P(x) \land Q(x)) \quad \text{premise} \\
2 & z \\
3 & P(z) \land Q(z) \\
4 & Q(z) \\
? & \forall x Q(x) \quad \forall x i 2–? \text{ where } \phi \text{ is } Q(x)
\end{array}
$$

Because we were careful to ensure that the variable $z$ is free in the premise, we can apply the rule $\forall x$ e to the premise. This gives an instance of the formula – that is, a substitution of variable in the formula – on Line 4.

$$
\begin{array}{c}
1 & \forall x (P(x) \land Q(x)) \quad \text{premise} \\
2 & z \\
3 & P(z) \land Q(z) \quad \forall x e 1 \\
4 & Q(z) \\
? & \forall x Q(x) \quad \forall x i 2–4 \text{ where } \phi \text{ is } Q(x)
\end{array}
$$

We see that the last line of the assumption box follows directly from Line 3. The proof is finished so we can fill in the missing line numbers.

$$
\begin{array}{c}
1 & \forall x (P(x) \land Q(x)) \quad \text{premise} \\
2 & z \\
3 & P(z) \land Q(z) \quad \forall x e 1 \\
4 & Q(z) \land i_2 4 \\
5 & \forall x Q(x) \quad \forall x i 2–4 \text{ where } \phi \text{ is } Q(x)
\end{array}
$$

\textit{Key Concept:} The $\forall x$ i rule uses “backward” logical reasoning. We open the $\forall x$ i “box” with a variable that is free for the quantified variables in both the useful preceding lines and in the formula we are trying to deduce. We write the “fresh” variable as an assumption at the top of the box and the substituted formula at the bottom of the box. This constrains our reasoning so that the $\forall x$ i rule is applied correctly.
**Existential Elimination**

This is used when we know an existentially quantified formula \( \exists x \phi \) and we want to deduce a formula \( \chi \). The idea is: we substitute a “fresh” variable, such as \( t \), for \( x \) in the formula \( \phi \) and deduce the formula \( \chi \). This works because, even though we do not know the exact value of \( t \), we can conclude that \( \chi \) is true.

In this course, when using existential elimination, we almost always are trying to deduce an existentially quantified formula. This line of reasoning is:

(a) From a previous line in the proof, we have \( \exists x \phi \)

(b) We want to deduce a different formula \( \chi \) that has the form \( \exists x \psi \)

(c) We use existential elimination to introduce a “fresh” variable \( t \); to do this, we need \( t \) to be free for \( x \) in \( \phi \) and free for \( x \) in \( \psi \). This “opens” an assumption box.
   (i) We substitute \( t \) for \( x \) in \( \phi \), which is written as \( \phi[t/x] \) but is easy to understand in an example
   (ii) We deduce a formula that has \( t \) in \( \psi \)
   (iii) We deduce \( \exists x \psi \). This “closes” the assumption box that we opened in step (c).

(d) We deduce \( \exists x \psi \).

There may be times when we are trying to deduce a formula \( \chi \) that is not existentially quantified. However, we most often encounter existential elimination when trying to deduce and existentially quantified formula.

**Example:** “Suppose that everything with property \( P \) also has property \( Q \), and that something has property \( P \). We can conclude that something has property \( Q \).

\[
\forall x (P(x) \rightarrow Q(x)), \exists x P(x) \vdash \exists x Q(x)
\]

We begin by writing the premises and conclusion.

<table>
<thead>
<tr>
<th></th>
<th>( \forall x (P(x) \rightarrow Q(x)) )</th>
<th>premise</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \exists x P(x) )</td>
<td>premise</td>
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<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>?</td>
<td>( \exists x Q(x) )</td>
<td></td>
</tr>
</tbody>
</table>

A useful guideline in predicate proofs is to try to replace every existentially quantified formula with a formula that has a “fresh” variable; this often produces formulas where we can apply the rules of natural deduction for propositions.
Applying this guideline to Line 2, we would open an assumption box for existential elimination. To do this, we must: introduce a fresh variable, such as $z$, that is free in Lines 1–2 and the conclusion; write the variable $z$ and $\phi[z/x]$ in the top of the box as an assumption; and put the formula we are trying to deduce at the bottom of the assumption box. We know that the conclusion is deduced using existential elimination, so we note that in the “rules” column.

1 $\forall x (P(x) \rightarrow Q(x))$ premise  
2 $\exists x P(x)$ premise  
3 $z P(z)$ assumption  
   $\vdots$  
   $\exists x Q(x)$  

$\exists x Q(x)$ $\exists x \in 2, 3?$

We examine the proof outline carefully. We observe that an easy way to deduce the formula at the bottom of the assumption box is to deduce it from an instance of the predicate $Q(\cdot)$; the only variable available is $z$ so we temporarily suppose that we have arrived at $\exists x Q(x)$ by existential introduction from the formula $Q(z)$. We add this formula to our proof and note the rule that we suppose we might be able to use.

1 $\forall x (P(x) \rightarrow Q(x))$ premise  
2 $\exists x P(x)$ premise  
3 $z P(z)$ assumption  
   $\vdots$  
   $Q(z)$  
   $\exists x Q(x)$ $\exists i ?$ where $\psi$ is $Q(x)$  

$\exists x Q(x)$ $\exists x \in 2, 3?$

Finally, we observe that the unused premise in Line 1 can be substituted using the variable $z$ to get $P(z) \rightarrow Q(z)$. This requires the use of universal elimination, so ensuring that the variable $z$ was free in all of the premises and in the conclusion was a good idea: it gave us the freedom to make this substitution.

The substituted premise, plus the first line of the assumption box, can be used to deduce $Q(z)$ by the $\rightarrow i$ rule. This completes the proof.
1. \[ \forall x (P(x) \rightarrow Q(x)) \] premise
2. \[ \exists x P(x) \] premise

3. \[ z \quad P(z) \] assumption
4. \[ P(z) \rightarrow Q(z) \] \( \forall x e 1 \)
5. \[ Q(z) \rightarrow e 4,3 \]
6. \[ \exists x Q(x) \] \( \exists i 5 \) where \( \psi \) is \( Q(x) \)

7. \[ \exists x Q(x) \] \( \exists x e 2, 3–6 \)

**Key Concept:** The \( \exists x e \) rule uses “forward” logical reasoning. When we open the \( \exists x e \) “box”, we put both a fresh variable and a substituted formula at the top of the box – these are the assumptions – and the formula that is right below the box inside as the last line of the box – this the desired conclusion. This constrains our reasoning so that the \( \exists x e \) rule is applied correctly.

**Caution:** The \( \exists x e \) guideline, which is that the formula to be deduced is existentially quantified, does not always apply. The textbook example on pages 115–116 shows how a predicate, that has a variable that is free for the existential quantifier being used, can be deduced using the \( \exists x e \) rule. In predicate logic, context is important.