CISC 204  Class 22

Review of Material

Test #3 in this course will cover proofs in predicate logic. The material will be from Week 6 and Week 7 of the course. The main concepts are:

- Universal quantification: introduction and elimination
- Existential quantification: introduction and elimination
- Equivalences: scope, commutation, negation, distribution
- Reasoning: using rules of inference in proofs

To study for this test, students should be able to do at minimum these exercises from the textbook:

§ 2.3: 4(c), 6(b), 7(a), 7(c), 9(a), 9(d), 9(n), 11(a), 11(c), 13(e), 13(h)

In addition, from Section 2.3 of the textbook, students should be able to prove the equivalences in Theorem 2.13 on Pages 117–118. Special attention should be given to Part 1 and Part 3, which are important concepts for later courses in computing.

The next two pages of reference material will be reproduced on the test. Students are expected to understand the rules and how to apply them.

After these pages, proofs are provided for theorems that we went over in class. There are subtle differences that students should be sure that they understand for this test.
Reference: Rules of Deduction in Propositional Logic

The basic rules in propositional logic:

<table>
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<tr>
<th>Introduction</th>
<th>Elimination</th>
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Some derived rules in propositional logic:

\[ \phi \rightarrow \psi \quad \neg \psi \]
\[ \neg \phi \quad \text{MT} \]
\[ \neg \neg \phi \quad \text{PBC} \]
\[ \phi \quad \neg \neg \phi \quad \text{LEM} \]
Reference: Rules of Deduction in Predicate Logic

The basic rules in predicate logic:

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<tr>
<th>( \forall x )</th>
<th>introduction</th>
<th>elimination</th>
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<tbody>
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</table>

\( \exists x \phi \)  | \( \exists x \phi \)  | \( \exists x \phi \)  |
\( \exists x \phi \)  | \( \vdash \exists x \phi \)  | \( \vdash \exists x \phi \)  |
\( \vdash \phi[t/x] \)  | \( \vdash \chi \)  | \( \vdash \chi \)  |

Substitution rules in predicate logic:

\( t = t \)  | \( t_1 = t_2 \)  | \( \phi[t_1/x] = e \)
\( =i \)  | \( =e \)  |
**Example:** Universal quantifiers commute and can be re-named

From the premise $\forall x \forall y P(x, y)$ we can deduce two conclusions: That the quantifiers commute, so $\vdash \forall y \forall x P(x, y)$, and that variables can be re-named, so $\vdash \forall y \forall x P(y, x)$.

One proof for commuting universal quantifiers is:

1. $\forall x \forall y P(x, y)$ premise
2. $w$
3. $\forall y P(z, y)$ $\forall x \text{ e 1}$
4. $P(z, w)$ $\forall x \text{ e 4}$
5. $\forall x P(x, w)$ $\forall x \text{ i 3–5}$
6. $\forall y \forall x P(x, y)$ $\forall x \text{ i 2–6}$

A proof for re-naming variables is:

1. $\forall x \forall y P(x, y)$ premise
2. $z$
3. $\forall y P(z, y)$ $\forall x \text{ e 1}$
4. $w$
5. $P(z, w)$ $\forall x \text{ e 3}$
6. $\forall x P(z, x)$ $\forall x \text{ i 4–5}$
7. $\forall y \forall x P(y, x)$ $\forall x \text{ i 2–6}$

The difference in these proofs is when the quantifier is eliminated. In the second proof, by eliminating it early, we can only re-name the variables. In the first proof, by first introducing a second fresh variable $w$ and next eliminating the quantifier, the quantifiers commute.

We can also demonstrate that existential quantifiers commute. One proof is:

1. $\exists x \exists y P(x, y)$ premise
2. $z$
3. $\exists y P(z, y)$ assumption
4. $w$
5. $P(z, w)$ assumption
6. $\exists x P(x, w)$ $\exists x \text{ e 3}$
7. $\exists y \exists x P(x, y)$ $\exists x \text{ i 3–5}$
8. $\exists y \exists x P(x, y)$ $\exists x \text{ e 1, 2–6}$

Students should understand that *introduction* is needed on Line 4 and Line 5, then we switch reasoning (to close the assumption boxes) and use *elimination* in Line 6 and Line 7.

In Line 3, the variables $z$ and $w$ are free in the formula $P(z, w)$ so we can choose the order in which we re-introduce the quantifiers. Line 4 is a key line: it is where the commutation happens, because we chose to re-introduce the quantified variable $x$ first.