CISC 365

Reduction to prove that vertex cover is NP complete

This illustrates the inventiveness used to make one problem do the work of another problem.
In this case, make vertex cover do the work of 3-SAT.

Background: SAT $\le_p$ 3-SAT

3-SAT is satisfiability with the restriction that every clause contains exactly 3 distinct literals.
3-SAT is NP complete. The reduction SAT $\le_p$ 3-SAT is given in pages 1082-1085 of the text. Introduce extra boolean variables and make the SAT formula longer to turn it into 3-SAT form.

example SAT formula $(a \vee b) \wedge (c \vee d \vee e \vee f \vee g)$

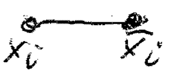
equivalent 3-SAT formula $(a \vee b \vee x_1) \wedge (a \vee b \vee \bar{x}_1) \wedge (c \vee d \vee \bar{y}_1)$
$\wedge (y_1 \vee e \vee \bar{y}_2) \wedge (y_2 \vee f \vee g)$

(A long clause with $k > 3$ literals turns into $k-2$ clauses of size 3, using $k-3$ new Boolean variables)

Prove 3-SAT $\le_p$ Vertex Cover

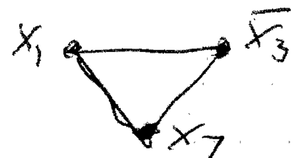
Make up "gadgets" in the graph to represent the literals and clauses in 3-SAT.
Finding the right gadgets takes insight and inspiration.
These gadgets work:
3-SAT input has $\boxed{n \text{ boolean variables and } c \text{ clauses}}$

1) Translate Boolean variable x_i to



This creates $\begin{cases} 2n \text{ vertices} \\ n \text{ edges} \end{cases}$

② Translate each clause to a triangle in the graph
eg $(x_1 \vee x_7 \vee \overline{x_3})$ becomes



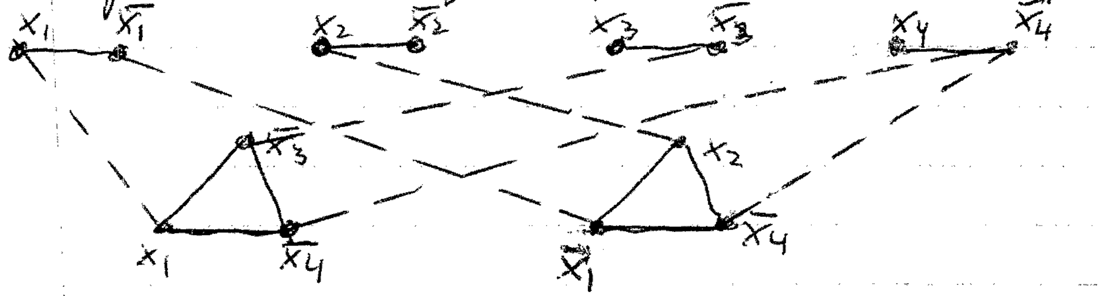
There are c triangles, so this creates $\begin{cases} 3c \text{ vertices} \\ 3c \text{ edges} \end{cases}$

③ Connect each vertex in a clause gadget to the same-labeled vertex in a boolean variable gadget

We have a total of $2n + 3c$ vertices
$n + 6c$ edges

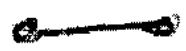
④ Does this graph have a vertex cover of size $n+2c$?
Yes iff the 3-SAT formula is satisfiable

Example $(x_1 \vee \overline{x_3} \vee \overline{x_4}) \wedge (\overline{x_1} \vee x_2 \vee \overline{x_4})$



n of these

c of these

The solid edges take exactly $n + 2c$ vertices to cover.
(Include at least one vertex from  and at least two vertices from )
There are $2^n \cdot 3^c$ covers of size $n+2c$, for the solid edges.

Now add the dashed edges ($3c$ of them). This graph can be covered with $n+2c$ vertices iff 3-sat formula is satisfiable.
Here, x_4 = false satisfies 3-SAT. Correspondingly, choose $\overline{x_4}$ in the boolean variable gadget. Choose the other two vertices in the clause gadgets.