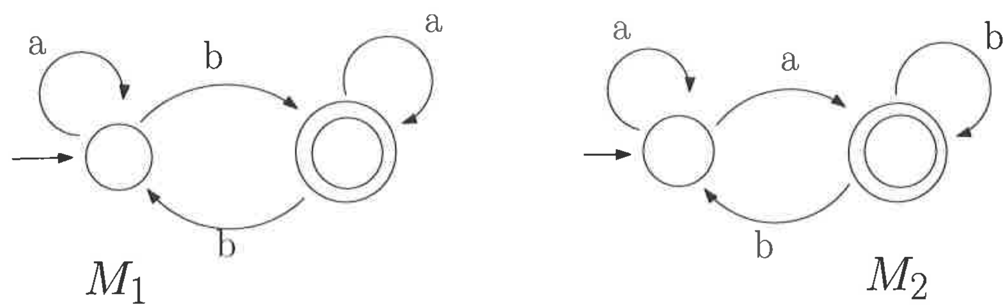


Figure 1: Finite automaton M_1 (on the left) and finite automaton M_2 (on the right).

1. Define:

$$A_{NFA} = \{ < M, w > \mid M \text{ is an NFA, } w \text{ is a string and } w \in L(M)\}$$

$$EQ_{DFA,REX} = \{ < M, R > \mid M \text{ is a DFA, } R \text{ is a regular expression, and } L(M) = L(R) \}.$$

Recall that DFA (respectively, NFA) stands for deterministic (respectively, nondeterministic) finite automaton. Let M_1 and M_2 be the finite automata given in Figure 1 and let R_1 be the regular expression $a^*b(a^*b)^*$

Answer the following questions and give <u>reasons</u> for your answers.

(i) Is $< M_1, aba > \in A_{NFA}$? Yes. M_1 accepts aba.

(ii) Is $< M_1, bab > \in A_{NFA}$? No. M_1 does not accept aba.

(iii) Is $< M_2, abb > \in A_{NFA}$? Yes. M_2 accepts abb.

(iv) Is $< M_2, abab > \in A_{NFA}$? Yes. M_2 accepts abab.

(v) Is $< M_1, R_1 > \in EQ_{DFA,REX}$? No. $bb \in L(R_1) - L(M_1)$

(vi) Is $< M_2, R_1 > \in EQ_{DFA,REX}$? No. M_2 is not a DFA

(vii) Is $< M_1, M_1 > \in EQ_{DFA,REX}$? No. The second input should be a regular expression.

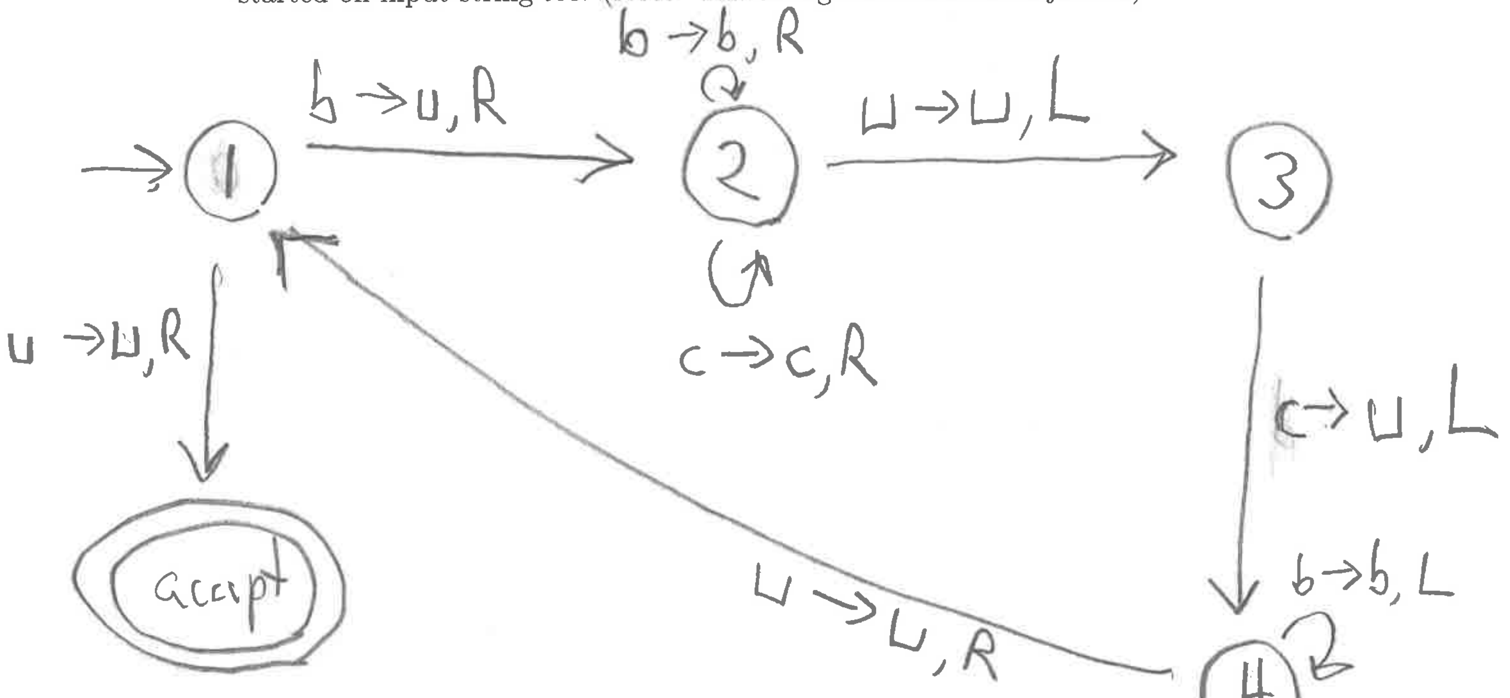
(viii) Is $< M_1, M_2 > \in EQ_{DFA,REX}$? No. ——— " ———

(ix) Is $< R_1, R_1 > \in EQ_{DFA,REX}$? No. The first input should be a DFA.

2. Give a complete construction, that is, a state transition diagram of a **single-tape deterministic Turing machine** M that decides the language

$$\{ b^i c^i \mid i \geq 0 \}$$

- Give also the sequence of configurations that your Turing machine M enters when started on input string bbc. (Note: The string bbc should be rejected.)



All undefined transitions go to the reject state. Sequence of configurations:

$1bbc$
$\sqcup 2bc$
$\sqcup b2c$
$\sqcup bc2\sqcup$
$\sqcup b3c\sqcup$
$\sqcup 4b\sqcup$
$4\sqcup b\sqcup$
$\sqcup 1b\sqcup$
$\sqcup\sqcup 2\sqcup$
$\sqcup 3\sqcup\sqcup$
reject

3. (i) (3 marks) Consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = (2 \cdot n) - 1$. (Here $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of positive integers.) Answer the following questions and give reasons for your answers.

(a) Is f onto? No. 2 is not an image of any element of $\mathbb{N}$.

(b) Is f one-to-one?

Yes. $2x-1 = 2y-1$ implies $x=y$.

(c) Is f a correspondence?

No because it is not onto.

(ii) (7 marks) Let $T = \{(i, j, k) \mid i, j, k \in \mathbb{N}\}$. Show that T is countable.

The following list contains all elements of T:

Stage 1: List triples (i,j,k) where $i+j+k=3$
(only $(1,1,1)$)

Stage 2: List triples (i,j,k) where $i+j+k=4$
$((1,1,2), (1,2,1), (2,1,1))$

.
.
.

Stage x: List triples (i,j,k) where $i+j+k = x+2$

.
.
.

4. Give an implementation-level description of a **deterministic** Turing machine M that decides the following language A over the alphabet $\Sigma = \{b, c\}$.

$$A = \{ \; w \in \Sigma^* \;\; | \;\; |w|_b = 2^{(|w|_c)} \; \}$$

Above the number of occurrences of symbol b (respectively, c) in a string w is denoted as $|w|_b$ (respectively, $|w|_c$). "$2^{(|w|_c)}$" denotes "2 to the power $|w|_c$".
If you wish, your deterministic TM M can use more than one tape.

The TM M uses 3 tapes.

M = "On input $w \in \Sigma^*$:
1. Write \$ to tape 2.
2. Repeat as long as input tape has unmarked c's:
 - Mark the next unmarked c on input tape.
 - Copy contents of tape 2 two times to tape 3.
 - Replace tape 2 contents by tape 3 contents and erase tape 3.

// After 2. finishes tape 2 will have $2^{|w|_c}$ symbols \$.

3. By scanning tape 1 and tape 2 in parallel, check that the number of b's on tape 1 equals the number of \$'s on tape 2. If yes, accept. If no reject."

5. (i) (4 marks) Does the following instance of the Post Correspondence Problem have a match (that is, a solution). Justify your answer.

$$\{ \left[\frac{baab}{ab}\right], \left[\frac{bab}{baab}\right], \left[\frac{baab}{baaab}\right], \left[\frac{ab}{abba}\right] \}$$

Match: $\left[\frac{ab}{abba}\right]\left[\frac{baab}{ab}\right]$

(ii) (6 marks) We define the following language

$PCP = \{< P > \mid P$ is an instance of the Post Correspondence Problem with a match $\}$

Answer the following questions and **justify** your answers.

(a) Does there exist a decidable language A such that $PCP \leq_m A$?

No. PCP is undecidable and an undecidable language cannot be mapping reduced to a decidable language.

(b) Does there exist an undecidable language B such that $PCP \leq_m B$?

Yes. $PCP \leq_m PCP$ via the identity function.

(c) Does there exist a decidable language C such that $C \leq_m PCP$?

Yes. Choose $C = \{b\}$ and let P_0 be the PCP instance from (i) and P_1 is the PCP instance $\{\frac{a}{ab}\}$. Define $f(b) = \langle P_0 \rangle$ and $f(x) = \langle P_1 \rangle$ when $x \neq b$. Function f is computable and $C \leq_m PCP$ via function f.

6. Let

$B_{TM} = \{ < M > \mid M$ is a deterministic Turing machine with input alphabet $\{a, b\}$, such that all strings accepted by M begin with b and $L(M) \neq \emptyset \}$.

Without using Rice's theorem show that B_{TM} is *undecidable.*

We reduce A_{TM} to B_{TM}. Suppose TM P decides B_{TM}. Construct a decider Q for A_{TM}:

Q = "On input $\langle M, w\rangle$ where M is a TM
1. Construct TM M_w with input alphabet $\{a, b\}$
   M_w = "On input x:
   1. If $x = b$, accept. Else continue.
   2. Run M on w and accept if M accepts"
2. Run P on $\langle M_w\rangle$. If P accepts, reject. If P rejects, accept"

Why this works:

- If M does not accept w, $L(M_w) = \{b\}$ and $\langle M_w\rangle \in B_{TM}$.
- If M accepts w, $L(M_w) = \{a, b\}^*$ and $\langle M_w\rangle \notin B_{TM}$.

7. (i) (4 marks) In each part circle the correct answer.

(a) $n^2 = O((\log n)^5 \cdot n)$ TRUE FALSE

(b) $n \cdot \log n = o(n^2)$ TRUE FALSE

(c) $2^n = o(3^n)$ TRUE FALSE

(d) $1 = O(n^2)$ TRUE FALSE

(ii) (6 marks) We define

$$\text{ALL}_{DFA} = \{ < A > \mid A \text{ is a DFA with alphabet } \Sigma \text{ and } L(A) = \Sigma^* \}.$$

Show that ALL_{DFA} is in P. In your solution you can assume known that the reachability problem on directed graphs (PATH) is in P.

ALL_{DFA} is decided by TM Q:

Q = "On input $\langle A \rangle$ where A is a DFA

1. Compute the set of states R of A that are reachable from the start state
2. If R contains a non-final state or a state with some transition undefined, reject.
3. If didn't reject yet, accept"

Q decides ALL_{DFA} because $L(A) \neq \Sigma^*$ iff some reachable state is non-final or has an undefined transition.

Step 1 can be done in polynomial time because PATH ∈ P.

Step 2 involves going through elements of R (subset of states of A) and is also polynomial time.

8. Give an example of a non-context-free language A such that A is in the class L (= SPACE($\log n$)). You should briefly explain how a logarithmic space deterministic TM decides your language A. You do not need to prove that A is non-context-free.

$A = \{a^i b^i c^i \mid i \geq 0\}$

Deterministic TM for A. Q = "On input w:

1. Scan w and check that it is in $a^*b^*c^*$
2. Return the tape head to left end.
3. Count the number of a's, b's, and c's, respectively, and store the values in binary on the work tape
4. By zig-zagging on the work tape, check that all the counter values are equal."

Q stores on work tape 3 binary numbers, each having length at most $\log |w|$.

9. Let B be NP-complete and assume that $B \leq_P \overline{B}$. (Here $\overline{B}$ is the complement of B and $\leq_P$ is polynomial time reducibility.)

Show that the above implies NP = coNP.

$B \leq_p \overline{B}$ implies $\overline{B} \leq_p B$ (via the same reduction function)

Let $A \in NP$ be arbitrary. Since B is NP-complete, we have $A \leq_p B$ which implies $\overline{A} \leq_p \overline{B}$. By transitivity of $\leq_p$, we have $\overline{A} \leq_p B$. Since $B \in NP$, it follows that $\overline{A} \in NP$, that is, $A \in coNP$. We have shown that $NP \subseteq coNP$.

Converse inclusion:

$A \in coNP$ implies $\overline{A} \in NP \subseteq coNP$ implies $A \in NP$.

shown above

10. Recall that $\leq_L$ denotes the logarithmic space reducibility. Answer the following questions and justify your answers. *Note:* To show that an implication does not hold, one should give a counter-example.

(i) Let B be an NP-complete language and A is a language such that $A \leq_L B$. Does this imply that A is in NP ?

Yes. $A \leq_L B$ implies $A \leq_p B$ and NP is closed under $\leq_p$.

(ii) Let B be a context-free language and A is a language such that $A \leq_L B$. Does this imply that A is in NP ?

Yes. Context-free languages are in $P \subseteq NP$ and NP is closed under $\leq_L$.

(iii) Let B be a context-free language and A is a language such that $B \leq_L A$. Does this imply that A is in NP ?

No. Let M_1 be a TM s.t. $L(M_1) = \emptyset$ and M_2 a TM s.t. $L(M_2) = \{b\}$. Define f by setting

$$f(x) = \begin{cases} \langle M_1 \rangle & \text{if } x = a \\ \langle M_2 \rangle & \text{if } x \neq a \end{cases}$$

f is computable in log-space and $\{a\} \leq_L E_{TM}$ via function f. E_{TM} is undecidable and, hence, not in NP.

11. What is the relationship (equal "="; strict inclusion "⊂" or "⊃"; inclusion that is not known to be strict "⊆" or "⊇") between the following pairs of complexity classes. **Justify** your answers.

(i) TIME(n^3) and TIME($n^3 \cdot \log n$) $TIME(n^3) \subseteq TIME(n^3 \cdot \log n)$

n^3 is not $o\left(\frac{n^3 \cdot \log n}{\log(n^3 \cdot \log n)}\right)$ and time hierarchy theorem does not apply.

(ii) SPACE($n^3 + n^2 \cdot (\log n)^3$) and SPACE($n^3 \cdot \log n$)

$SPACE(n^3 + n^2 \cdot (\log n)^3) \subsetneq SPACE(n^3 \cdot \log n)$ because $n^3 + n^2 (\log n)^3 = o(n^3 \cdot \log n)$ and use space hierarchy theorem.

(iii) SPACE(2^n) and SPACE(2^{n+5})

$SPACE(2^n) = SPACE(2^{n+5})$ because

$2^{n+5} = 32 \cdot 2^n$

(iv) TIME(2^n) and TIME(3^n) $TIME(2^n) \subsetneq TIME(3^n)$

$\frac{3^n}{\log 3^n} = \frac{3^n}{(\log 3) \cdot n}$ and $\lim_{n \to \infty} 2^n \Big/ \frac{3^n}{(\log 3) \cdot n} = 0$ and time hierarchy theorem gives strict inclusion

(v) SPACE(n^5) and NSPACE($n^2 \cdot \log n$)

$NSPACE(n^2 \cdot \log n) \underset{\text{Savitch}}{\subseteq} SPACE(n^4 \cdot (\log n)^2) \underset{\text{space hierarchy theorem}}{\subsetneq} SPACE(n^5)$