INSTRUCTIONS

- **Aids allowed:** You may bring in one 8.5 × 11 inch sheet of paper containing notes, and use it during the exam. The sheet can be written/printed on both sides.
- This examination is THREE HOURS in length.
- **Answer each question in the space provided** (on the question paper). There is an extra page at the end of the exam, if more space is needed. **Please write legibly.**
- Each of the 11 questions is worth 10 marks. The exam is marked out of 100 possible marks, and your mark is calculated as the sum of your 10 best answers. You can answer all 11 questions and only the 10 best answers are included in the mark.

PLEASE NOTE: Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written.

**STUDENT NUMBER:**

Please write extremely carefully.

| Problem 1  | /10 |
| Problem 2  | /10 |
| Problem 3  | /10 |
| Problem 4  | /10 |
| Problem 5  | /10 |
| Problem 6  | /10 |
| Problem 7  | /10 |
| Problem 8  | /10 |
| Problem 9  | /10 |
| Problem 10 | /10 |
| Problem 11 | /10 |
| Total*     | /100 |

*The total mark is calculated as the sum of your 10 best answers.

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1. Define:

$$A_{DFA} = \{ < M, w > \mid M \text{ is a DFA, } w \text{ is a string and } w \in L(M) \}$$

$$EQ_{DFA} = \{ < M, N > \mid M \text{ and } N \text{ are DFAs, and } L(M) = L(N) \}$$

$$INCL_{DFA} = \{ < M, N > \mid M \text{ and } N \text{ are DFAs, and } L(M) \subseteq L(N) \}$$

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Figure 1: (i) DFA $A$ (on left); and (ii) DFA $B$ (on right).

Answer the following questions and give reasons for your answers.

(i) Is $< A, aab > \in A_{DFA}$?

(ii) Is $< A, abb > \in A_{DFA}$?

(iii) Is $< B, bbaabbaab > \in A_{DFA}$?

(iv) Is $< A, B > \in EQ_{DFA}$?

(v) Is $< A, A > \in EQ_{DFA}$?

(vi) Is $< B, A > \in EQ_{DFA}$?

(vii) Is $< A, B > \in INCL_{DFA}$?

(viii) Is $< B, A > \in INCL_{DFA}$?

(ix) Is $< A, A > \in INCL_{DFA}$?
2. (i) Consider the language

\[ C = \{ < M, w > \mid M \text{ is a DFA, and some string of } L(M) \text{ contains string } w \text{ as a substring } \} \].

Is the language \( C \) decidable or undecidable? Prove your answer.

(ii) Consider the language

\[ D = \{ < M > \mid M \text{ is a Turing machine and there exists a DFA } A \text{ such that } L(M) \subseteq L(A) \} \].

Is the language \( D \) decidable or undecidable? Prove your answer.
3. (i) Does the following instance of the Post Correspondence Problem have a match (that is, a solution). Justify your answer.

\[
\{ \begin{array}{c}
\left[ \begin{array}{c}
1001 \\
01
\end{array} \right], \\
\left[ \begin{array}{c}
101 \\
1001
\end{array} \right], \\
\left[ \begin{array}{c}
1001 \\
10001
\end{array} \right], \\
\left[ \begin{array}{c}
01 \\
0110
\end{array} \right]
\end{array} \} 
\]

(ii) Let \( \Sigma = \{ a, b \} \). We denote \( \text{finite}(\Sigma) = \{ L \subseteq \Sigma^* \mid L \text{ is finite} \} \). The elements of the set \( \text{finite}(\Sigma) \) are the finite languages over \( \Sigma \).

Show that the set \( \text{finite}(\Sigma) \) is \textit{countable}. 
4. Give a complete construction, that is, a state transition diagram of a single-tape deterministic Turing machine $M$ that decides the language

$$\{ a^i b^k \mid 0 \leq i \leq k \}$$

- Give also the sequence of configurations that your Turing machine $M$ enters when started on input string $abb$. 
5. Given an example of a non-context-free language $A$ such that $A$ is in the class $L$ ($= \text{SPACE}(\log n)$). You should briefly explain how a logarithmic space deterministic TM decides your language $A$. You do not need to prove that $A$ is non-context-free.

6. Let $B$ be a Turing-recognizable language and assume that $B \leq_m \overline{B}$. ($\overline{B}$ is the complement of $B$.)

Prove that $B$ is decidable.
7. What is the relationship (equal “=”; strict inclusion “⊂” or “⊃”; inclusion that is not known to be strict “⊆” or “⊇”) between the following pairs of complexity classes. 

**Justify** your answers.

(i) \( \text{TIME}(n^3 \cdot \log n + n \cdot (\log n)^2) \) and \( \text{TIME}(n^3 + n^2 \cdot (\log n)^5) \)

(ii) \( \text{SPACE}(2^{n^2}) \) and \( \text{SPACE}(2^{n^2+n}) \)

(iii) \( \text{TIME}(2^{n^2}) \) and \( \text{TIME}(n^3 \cdot 2^n) \)

(iv) \( \text{NSPACE}(n^2 \cdot (\log n)^3) \) and \( \text{SPACE}(n^5) \)

(v) \( \text{NTIME}(n^2 \cdot \log n) \) and \( \text{SPACE}(n^5) \)
8. Define
   \[ E_{CFG} = \{ < G > \mid G \text{ is a context-free grammar and } L(G) = \emptyset \} \].

   Answer the following questions and justify your answers.

   (i) Does there exist a decidable language \( A \) such that \( A \leq_m E_{CFG} \)?

   (ii) Does there exist a decidable language \( B \) such that \( E_{CFG} \leq_m B \)?

   (iii) Does there exist an undecidable language \( C \) such that \( C \leq_m E_{CFG} \)?

   (iv) Does there exist an undecidable language \( D \) such that \( E_{CFG} \leq_m D \)?
9. We define

\[ P_{TM} = \{ < M, w > \mid M \text{ is a Turing machine and } w \text{ is a string and } M \text{ accepts } w^k \text{ for all positive integers } k \}. \]

**Without using** Rice’s theorem show that \( P_{TM} \) is **undecidable**.
10. What is the relationship between the following pairs of classes of languages. In each case **insert the correct symbol** on the dotted line between each pair: equal (“=”); strict inclusion in one direction (“STRICT ⊂” or “⊃ STR”) inclusion that is not known to be strict (“⊆” or “⊇”); or incomparable/not known (“OTHER”). The last option includes cases where the classes are incomparable or no inclusion relation is known.

- In order to avoid ambiguity between strict and non-strict inclusions, please write “STRICT ⊂” or “⊃ STR” for the strict inclusions.

In addition to standard notations for complexity classes from our textbook, we denote:

- **REGULAR** = the class of regular languages
- **LBA** = the class of languages decided by deterministic linear bounded automata

You do not need to explain the answers.

(i) NP ...................................... PSPACE

(ii) REGULAR ................................ NL

(iii) L ........................................ NP

(iv) P ........................................ PSPACE

(v) LBA .................................... REGULAR

(vi) LBA .................................... SPACE(n)

(vii) LBA ..................................... NL

(viii) coNP ................................. NP

(ix) coNL ................................ NL

(x) coNL ................................ NP
11. Here $\leq_P$ denotes the polynomial time reducibility relation. Note: To show that a particular implication does not hold, you need to give a counter-example.

(i) If $A \leq_P B$ and $B$ is a regular language, does that imply that $A$ is in $P$?
   Justify your answer: why or why not?

(ii) If $A \leq_P B$ and $B$ is a regular language, does that imply that $A$ is regular?
    Justify your answer: why or why not?

(iii) If $A \leq_P B$ and $A$ is a regular language, does that imply that $B$ is context-free?
     Justify your answer: why or why not?

(iv) If $A \leq_P B$ and $A$ is a regular language, does that imply that $B$ is decidable?
     Justify your answer: why or why not?
Extra page.