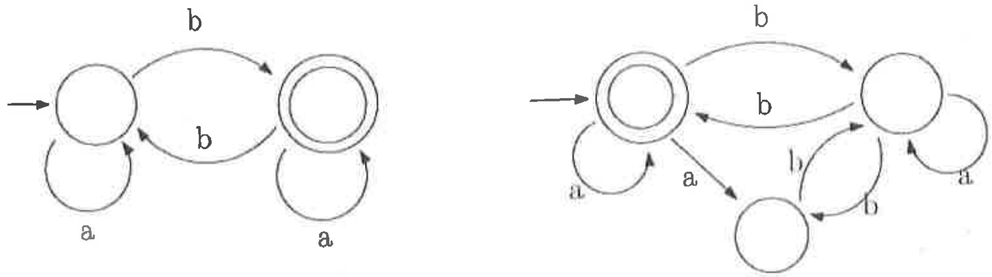

Figure 1: A finite automaton M_1 (on left) and a finite automaton M_2 (on right).

1. Define:

$$A_{NFA} = \{ < B, w > \mid B \text{ is an NFA that accepts input string } w\},$$
$$\mathrm{EQ}_{DFA} = \{ < B, C > \mid B, C \text{ are DFAs and } L(B) = L(C) \}.$$

Recall that DFA (respectively, NFA) stands for deterministic (respectively, nondeterministic) finite automaton. Answer the following questions (for M_1 and M_2 given in the above figure) and give reasons for your asnwers.

(a) Is $< M_1, babb > \in A_{NFA}$? Yes. M_1 accepts babb.

(b) Is $< M_1, abba > \in A_{NFA}$? No. M_1 does not accept abba.

(c) Is $< M_2, abb > \in A_{NFA}$? Yes. M_2 accepts abb.

(d) Is $< M_2, aaabb > \in A_{NFA}$? Yes. M_2 accepts aaabb.

(e) Is $< M_1, M_2 > \in \mathrm{EQ}_{DFA}$? No. M_2 is not a DFA.

(f) Is $< M_1, M_1 > \in \mathrm{EQ}_{DFA}$? Yes. $L(M_1) = L(M_1)$.

(g) Is $< M_2, M_2 > \in \mathrm{EQ}_{DFA}$? No. M_2 is not a DFA.

2. (a) Consider the language

$$C = \{< M, w > \mid M \text{ is a DFA, and some string of } L(M) \text{ contains string } w \text{ as a substring } \}.$$

Is the language C decidable or undecidable? Prove your answer.

Decider for C:

P = "On input $\langle M, w \rangle$
1. Construct a DFA A for $L(M) \cap \Sigma^* w \Sigma^*$, where Σ is the alphabet of M. This is possible because regular languages are closed under intersection.
2. Decide whether $L(A) = \emptyset$ (DFA emptiness is decidable) If yes, reject. If no accept "

(b) Consider the language

$$D = \{< M > \mid M \text{ is a Turing machine and there exists a DFA } A \text{ such that } L(M) = L(A)\}.$$

Is the language D decidable or undecidable? Prove your answer.

D is undecidable. Two justifications (only one needed).

i) D is a non-trivial semantic property. Undecidability follows from Rice's theorem.

ii) D encodes the problem of deciding whether the language recognized by a TM is regular. This is shown in the text to be undecidable (in Theorem 5.3)

3. Give an implementation-level description of a **deterministic** one tape Turing machine that decides the following language A over the alphabet $\Sigma = \{c, d\}$. The number of occurrences of symbol c (respectively, d) in a string w is denoted $|w|_c$ (respectively, $|w|_d$).

$$A = \{\ w \in \Sigma^* \mid |w|_c \leq |w|_d \leq 2 \cdot |w|_c\ \}.$$

That is, in strings of A the number of occurrences of d is at least the number of occurrences of c and at most 2 times the number of occurrences of c.

P = "On input $w \in \Sigma^*$:

1. Repeat the following as long as both $\underbrace{\text{c's}}_{\text{unmarked}}$ and $\underbrace{\text{d's}}_{\text{unmarked}}$ remain:
   - scan the tape and mark one c and one d.
2. If after 1. there remain $\underbrace{\text{c's}}_{\text{unmarked}}$ on the tape reject.
   Else continue from 3.
3. Unmark all c's and d's.
4. Repeat the following as long as both unmarked c's and unmarked d's remain:
   - scan the tape and mark two d's and one c
5. If after the last scan unmarked d's remain, reject.
   Else accept /* this includes the case where the last scan could mark only one d */

4. Let

$$\text{TWO}_{\text{TM}} = \{ <M> \mid M \text{ is a deterministic Turing machine and } L(M) \text{ consists of exactly two strings} \}.$$

Without using Rice's theorem show that TWO_{TM} is *undecidable*.

We reduce A_{TM} to TWO_{TM}. Suppose TM P decides TWO_{TM}. Construct decider Q for A_{TM}:

Q = "On input $\langle M, w \rangle$

1. Construct decider TM N:

 N = "On input x:

 1. If $x = a$ or $x = aa$ $(a \in \Sigma)$, accept.
 2. Simulate M on w. If M accepts, accept. If M rejects, reject."

2. Run P on $\langle N \rangle$. If P accepts, reject. If P rejects, accept"

Explanation why this works:

If $\langle M, w \rangle \notin A_{TM}$, then $L(N) = \{a, aa\}$ and $\langle N \rangle \in TWO_{TM}$.

If $\langle M, w \rangle \in A_{TM}$, then $L(N) = \Sigma^*$ and $\langle N \rangle \notin TWO_{TM}$.

5

(a) (7 marks) Let $T = \{ (i, k, m) \mid i, k, m \in \mathbb{N} \}$.
<u>Show</u> that the set T is countable.

All elements of T can be listed as follows:

1. List (1, 1, 1)
2. List triples (i, k, m) where i+k+m = 4
3. List triples (i, k, m) where i+k+m = 5

⋮

Stage x : List triples (i, k, m) where i+k+m = x+2

All elements of T appear on above list.

(b) (3 marks) Are the following sets countable? For each case circle the correct answer – no explanation needed. If you circle both YES and NO, it is considered a wrong answer.

- The set $\{0, 1\}^*$. YES NO
- The set of all subsets of $\{0, 1\}^*$. YES NO
- The set of all <u>finite</u> subsets of $\{0, 1\}^*$. YES NO