Continuous Coordinate Transformations: Lecture 3

Spherical Displacements

In robotics, computer graphics, biomechanics and many other applications, it is common to represent a spherical displacement as successive rotations about principal axes of a coordinate frame. When consulting work in these other fields, it is important to observe that, although these rotations are frequently called “Euler angles”, the latter term can refer to a wide variety of conventions and will thus not be used in this course.

The simplest principal rotation is about the $Z$ axis. It is the spatial extension of the planar rotational displacement, but operates on spatial points and vectors rather than on planar objects. Because it is a rotation about $Z$, the $Z$ component of a point or vector must be unchanged; this means that the last row of the matrix must be the row vector $[0 0 1]$. For the matrix to be orthogonal the final column must also be the $Z$ elementary vector.

$$R_z(\phi) = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can similarly reason that rotation about $X$ has a closely related structure.

$$R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi) & \cos(\phi) \end{bmatrix}$$

The third rotation, about $Y$, also has a straightforward structure. Note that the order of the coordinate axes is reversed, and so is the “sense” of the rotation.

$$R_y(\phi) = \begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) \end{bmatrix}$$

When considering applying these primary rotations successively, we will use frames. For example, to rotate about $\vec{x}$ by $\phi_x$ (from $\{1\}$ to $\{2\}$) and then about $\vec{y}$ by $\phi_y$ (from $\{2\}$ to $\{3\}$) is

$$^3_2R_y(\phi_y)^2_1R_x(\phi_x) = ^3_1R$$
3.1 Principal-Axis Rotation Conventions

There are two primary axis conventions widely used to describe rotation. For vehicles, the rotations are described in terms of roll, pitch and yaw:

**Roll** refers to rotation about the long axis of the vehicle. This is the bow-stern axis on a boat, the nose-tail axis on a plane, or the hood-trunk axis on a car.

**Pitch** refers to rotation about the transverse axis of the vehicle, such that the front end rises and falls. This would be the starboard-port axis on a boat and the left-right axis on a plane(from wingtip to wingtip) or a car (from door to door).

**Yaw** refers to rotation about the vertical axis (usually coincident with the gravitational vector) of the vehicle.

When applying these primary rotations successively, we will use frames. For example, to rotate about \( \vec{x} \) by \( \phi_x \) (from \( \{1\} \) to \( \{2\} \)) and then about \( \vec{y} \) by \( \phi_y \) (from \( \{2\} \) to \( \{3\} \)) is

\[
3_2 \begin{bmatrix} R_y(\phi_y) & R_x(\phi_x) \end{bmatrix} = 3_1 R
\]

The most common orders of these principal rotations are \( XYZ \) and \( ZYX \). They can, however, be performed in \( 3! = 6 \) different orders:

\[ XYZ \quad XZY \quad YXZ \quad YZX \quad ZXY \quad ZYX \]

The other convention more naturally describes a robotic wrist. It “re-uses” a primary axis as its third rotation. The most example, called the \( ZYZ \) convention, is given in Equation 3.1.

\[
4_3 R_z \begin{bmatrix} R_y & R_z \end{bmatrix} = 4_1 R \tag{3.1}
\]

This convention is the most common, but this type of primary rotation can be performed in \( 3! = 6 \) different orders:

\[ XYY \quad XZX \quad YXX \quad ZYX \quad YZX \quad ZYZ \]

So, there are 12 possible conventions for using 3 primary rotations to rotate a column vector\(^\ast\). It is therefore not sufficient to provide 3 angles: an additional piece of information is needed, in this case to identify which axis convention is to be used.

3.2 Orthogonal Rotation Matrices and Spherical Displacements

We know that the principal-axis rotations are orthogonal, because of how we constructed them above. Also, we have proved that the product of two orthogonal matrices is itself orthogonal. It is possible to prove the converse, which is that we are always capable of factoring any \( R \) (3 × 3 orthogonal matrix) into a product of 3 principal-axis rotations\(^\dagger\).

\(^\ast\)There are 24 possible conventions if the vector could be a row or a column.

\(^\dagger\)Further details are in the hand-out on *Basic Coordinate Transformations*. 

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3.3 Cayley’s Formula

Let us consider the spherical displacement \( \vec{q} = R\vec{p} \). It is possible to show that \([R + I] \) is invertible provided that \( \vec{q} \neq -\vec{p} \). On this understanding,

\[
\vec{q} - \vec{p} = R\vec{p} - \vec{p} = S(\vec{q} + \vec{p})
\]

where \( S = [R - I][R + I]^{-1} \). We can now solve this equation for \( R \) as

\[
R = [I - S]^{-1}[I + S]
\]

Equation 3.2 is *Cayley’s formula* for a spherical displacement. By working with an arbitrary vector \( \vec{u} \), we can show that the matrix \( S \) is a *skew-symmetric* matrix, where \( S^T = -S \):

\[
S = \begin{bmatrix}
0 & -s^z & s^y \\
s^z & 0 & -s^x \\
-s^y & s^x & 0
\end{bmatrix}
\]

Equation 3.3

\( S \) takes any arbitrary vector \( \vec{u} \) and creates a vector orthogonal to \( \vec{u} \) as

\[
S\vec{u} = \begin{bmatrix}
0 & -s^z & s^y \\
s^z & 0 & -s^x \\
-s^y & s^x & 0
\end{bmatrix} \begin{bmatrix}
u^x \\
u^y \\
u^z
\end{bmatrix} = \vec{s} \times \vec{u}
\]

This shows that a cross product can be represented as a matrix product, and the matrix product can be converted into a rotation matrix: there is a deep relationship between spherical displacements and vector cross products. Some other properties \( A \) are:

- \( S \) is skew-symmetric, so \( S^T = -S \).
- Multiplied by itself, it is symmetric: \( S^2 = [S^2]^T \).
- Taking the third power: \( S^3 = -\|\vec{s}\|^2 S = (\vec{s} \cdot \vec{s})S^T \)

We will find several uses for a special case of the skew-symmetric matrix, when the underlying vector is of unit length. If \( \vec{k} = \vec{s}/\|\vec{s}\| \), then by expansion

\[
K^2 = \begin{bmatrix}
-(k^y)^2 - (k^z)^2 & k^x k^y & k^x k^z \\
k^x k^y & -(k^x)^2 - (k^z)^2 & k^y k^z \\
k^x k^z & k^y k^z & -(k^x)^2 - (k^y)^2
\end{bmatrix}
\]

Equation 3.4

A closely related matrix is the outer product of \( \vec{k} \), which is

\[
\vec{k}\vec{k}^T = \begin{bmatrix}
k^x k^x & k^x k^y & k^x k^z \\
k^x k^y & k^y k^y & k^y k^z \\
k^x k^z & k^y k^z & k^z k^z
\end{bmatrix}
\]

(3.5)
Equation 3.4 and Equation 3.5 are closely related:

\[ K^2 = \vec{k} \vec{k}^T - I \]  

(3.6)

Using trigonometry and an expansion of the cross-product matrix, we can normalize the vector \( \vec{s} \) as \( \vec{k} = \vec{s}/\|\vec{s}\| \) to find that

\[
\vec{s} = \tan\left(\frac{\phi}{2}\right) \vec{k} \\
\Rightarrow S = \tan\left(\frac{\phi}{2}\right) K
\]

(3.7)

where \( \phi \) and \( \vec{k} \) are the **Rodrigues parameters**.

In summary, given an orthogonal matrix \( R \) that represents a spherical displacement, we can find a vector \( \vec{k} \) and an angle \( \phi \).

We can substitute the Rodrigues parameters of Equation 3.7 into Cayley’s formula of Equation 3.2 and, after some trigonometric and algebraic simplification, express a rotation matrix \( R \) as

\[
R = I + \sin(\phi)K + (1 - \cos(\phi))K^2
\]

(3.8)

Another way to express \( R \) is in terms of the outer product of \( \vec{k} \). Expanding Equation 3.6 into Equation 3.8 gives

\[
R = \cos(\phi)I + \sin(\phi)K + (1 - \cos(\phi))\vec{k} \vec{k}^T
\]

(3.9)

This means that, given a unit axis \( \vec{k} \) and angle \( \phi \), we can use either Equation 3.8 or Equation 3.9 to construct an orthogonal matrix that represents a spherical displacement.

A direct way find the axis from an orthogonal matrix is from the skew-symmetric matrix \( A = R - R^T \). We can expand Equation 3.8 into \( A \); to simplify, use the properties of a skew-symmetric matrix we made at the beginning of the lecture. Then

\[
A = R - R^T = 2 \sin(\phi) K
\]

(3.10)

Equation 3.10 means that, given \( R \), we can:

- compute the skew-symmetric \( A = R - R^T \)
- extract the vector \( \vec{a} \) from \( A \)
- find the angle of rotation \( \phi = \sin^{-1}(\|\vec{a}\|/2) \)
- find the axis of rotation \( \vec{k} = \vec{a}/\|\vec{a}\| \)
3.4 Summary of Spherical Displacements

In this lecture we have made some useful observations about the specification of a rotation matrix. If we use principal-axis matrices, we need 3 angles plus information about which convention to use. If we use an axis and an angle, we need 4 values (3 for the vector and 1 for the angle) plus a constraint on the vector, which is that it must be non-zero and of unit length; furthermore, from Equation 3.8, negating both the vector and the angle produce the same rotation matrix.

This suggests that there is a deep meaning to spherical displacements. When we look at the topology, 3D rotations will correspond to one of the most common yet perplexing low-dimension manifolds in mathematics.

**Corresponding Notes for Basic Coordinate Transformations:** pages 30–47