20.1 2-DOF Spherical $RR$ Manipulator

For a $RR$ manipulator that does not have parallel revolute axes, the simplest kind is when the axes are orthogonal. This kind has two cases: the axes intersect or they are skew. Intersecting orthogonal axes give rise to a spatial manipulator whose tip is confined to the surface of a sphere.

The first displacement is rotation about the global $Z$ axis. The second displacement is, by convention, rotation about the negative $Y$ axis of frame $\{2\}$, which for a positive angle $\theta$ takes the $X$ axis up towards the positive $Z$ axis; finally, the global origin is translated along the new $X$ axis by a non-negative distance $l_1$. The transformation, in vector notation, is shown in Figure 20.1.

![Figure 20.1: Spherical $RR$ manipulator. The effect is to rotate a vector $(l_1, 0, 0)$ first by $\eta$ about the global $Z$ axis and next by $\theta$ about the frame $\{1\}$ negative $Y$ axis.](image)

The displacement is done in the same manner as for parallel-axis planar manipulators. The joint vector is the same as the planar $RR$ manipulator, $\vec{\Upsilon}$ from Equation 7.4. The orien-
The orientation of the tip is

\[
R_{(Z,-Y)}(\bar{\Upsilon}) = [\hat{R}_Z(\eta)][\hat{R}_{(-Y)}(\theta)]
\]

\[
= \begin{bmatrix}
\cos(\eta) & -\sin(\eta) & 0 \\
\sin(\eta) & \cos(\eta) & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\cos(\theta) & 0 & -\sin(\theta) \\
0 & 1 & 0 \\
\sin(\theta) & 0 & \cos(\theta)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\cos(\eta) \cos(\theta) & -\sin(\eta) \cos(\theta) & -\cos(\eta) \sin(\theta) \\
\sin(\eta) \cos(\theta) & \cos(\eta) \cos(\theta) & -\sin(\eta) \sin(\theta) \\
\sin(\theta) & 0 & \cos(\theta)
\end{bmatrix}
\] (20.1)

so the displacement is Equation 20.1 followed by a translation. It is, in homogeneous notation,

\[
^{2}_{G^T} = [\hat{R}_{(Z,-Y)}(\bar{\Upsilon})][\hat{D}(\bar{x}, l_1)]
\]

\[
= \begin{bmatrix}
l_1 \cos(\eta) \cos(\theta) \\
l_1 \sin(\eta) \cos(\theta) \\
l_1 \sin(\theta)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\hat{R}_{(Z,-Y)}(\bar{\Upsilon}) & l_1 \cos(\eta) \cos(\theta) \\
l_1 \sin(\eta) \cos(\theta) & l_1 \sin(\theta) \\
\bar{\Upsilon}^T & 1
\end{bmatrix}
\] (20.2)

The translational part of Equation 20.2 is the familiar parametric equation of a sphere, often derived as the surface of revolution of a half-circle. The displacement of Equation 20.2 is shown graphically in Figure 20.2.

**Figure 20.2:** Spherical RR manipulator. Frame \{G\} is displaced to frame \{1\} by rotation about the global Z axis by angle \(\eta\); next, the frame is rotated about the frame \{1\} negative Y axis by angle \(\theta\) and translated along the newly rotated X axis by distance \(l_1\). The orientation of the tip of the manipulator is determined by its location.
The spherical $RR$ manipulator has a straightforward description as

<table>
<thead>
<tr>
<th>Type</th>
<th>Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-vector</td>
<td>$\vec{\Upsilon}_2$</td>
</tr>
<tr>
<td>parametric</td>
<td>$(\frac{C}{2} \hat{T}_x(\vec{\Upsilon}_2), \frac{C}{2} \hat{T}_y(\vec{\Upsilon}_2), \frac{C}{2} \hat{T}_z(\vec{\Upsilon}_2))$</td>
</tr>
<tr>
<td>implicit</td>
<td>$x^2 + y^2 + z^2 = l_1^2$</td>
</tr>
</tbody>
</table>

The workspace, defined by Equation 20.2, is depicted in Figure 20.3 as an ordinary sphere. Note that the definition provides an infinite number of ways of expressing the poles on the $Z$ axis, because for $\theta = \pm \pi/2$ any value of $\eta$ will specify the same vector.

![Figure 20.3: Spherical $RR$ manipulator. The workspace is the surface of a 2-sphere in 3-space, with angle $\eta$ measuring “longitude” and $\theta$ measuring “latitude”. The poles are unique vectors but do not have unique parameterizations for this manipulator.](image)

### 20.2 2-DOF Right Spatial $RR$ Manipulator

The final $RR$ manipulator we will study has orthogonal revolute axes that are skew, i.e., that do not intersect. There are many ways to construct such a manipulator so we will choose a simple method. The method is basically that of the planar $RR$ manipulator, but with the second rotation “out-of-plane”. Because the axes are orthogonal, this can be called a “right” spatial $RR$ manipulator.

The first displacement is a rotation about the global $Z$ axis followed by a translation along the global $X$ axis by positive distance $l_1$, taking the global frame to frame $\{1\}$. The second displacement is a rotation about the negative $Y$ axis of frame $\{1\}$ followed by a translation along the rotated $X$ axis by positive distance $l_2$. The vector version is shown in Figure 20.4.
The displacement equation is constructed using the same technique as was used for the spherical RR manipulator, with an extra translation. In homogeneous coordinates it is

\[
\hat{T}(\vec{Y}) = [\hat{R}_Z(\eta)][\hat{D}_X(l_1)][\hat{R}_{(-Y)}(\theta)][\hat{D}_X(l_2)]
\]

\[
= \begin{bmatrix}
\cos(\eta) & -\sin(\eta) & 0 & l_1 \cos(\eta) \\
\sin(\eta) & \cos(\eta) & 0 & l_1 \sin(\eta) \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\cos(\theta) & 0 & -\sin(\theta) & l_2 \cos(\theta) \\
0 & 1 & 0 & 0 \\
\sin(\theta) & 0 & \cos(\theta) & l_2 \sin(\theta) \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\hat{R}_{(Z,-Y)}(\vec{Y}_2) \\
l_1 \cos(\eta) + l_2 \cos(\eta) \cos(\theta) \\
l_1 \sin(\eta) + l_2 \sin(\eta) \cos(\theta) \\
l_2 \sin(\theta)
\end{bmatrix}
\]

(20.3)

The configuration of the right spatial RR manipulator is rather complicated. The vector and parametric notations look like those of the spherical RR manipulator, but have translational terms embedded in them. The implicit equation is a quartic, or fourth-order, surface.

<table>
<thead>
<tr>
<th>Type</th>
<th>Form</th>
</tr>
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<tbody>
<tr>
<td>2-vector</td>
<td>$\vec{Y}_2$</td>
</tr>
<tr>
<td>parametric</td>
<td>($\frac{G}{2}\hat{T}_x(\vec{Y}_2), \frac{G}{2}\hat{T}_y(\vec{Y}_2), \frac{G}{2}\hat{T}_z(\vec{Y}_2)$)</td>
</tr>
<tr>
<td>implicit</td>
<td>($x^2 + y^2 + z^2 + l_1^2 - l_2^2)^2 = 4l_1^2(x^2 + y^2)$</td>
</tr>
</tbody>
</table>

The configuration space of the simple spatial RR manipulator is a torus, as shown in Figure 20.6.*

*This public-domain image was from Wikimedia Commons, captured in November 2010.
Figure 20.5: Right spatial \textit{RR} manipulator. Frame $\{G\}$ is displaced to frame $\{2\}$. The orientation of the tip is determined by its location.

Figure 20.6: Right spatial \textit{RR} manipulator. The configuration space is a 2-torus in 3-space.