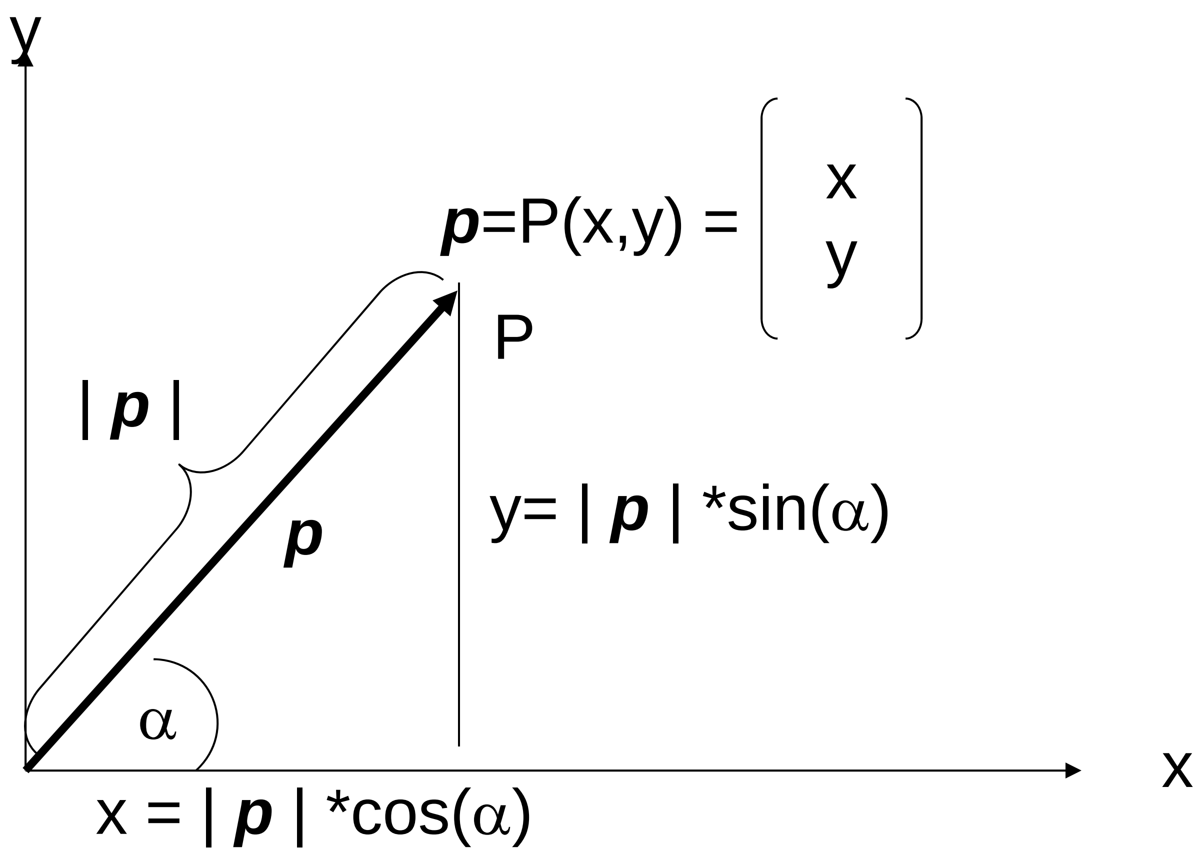
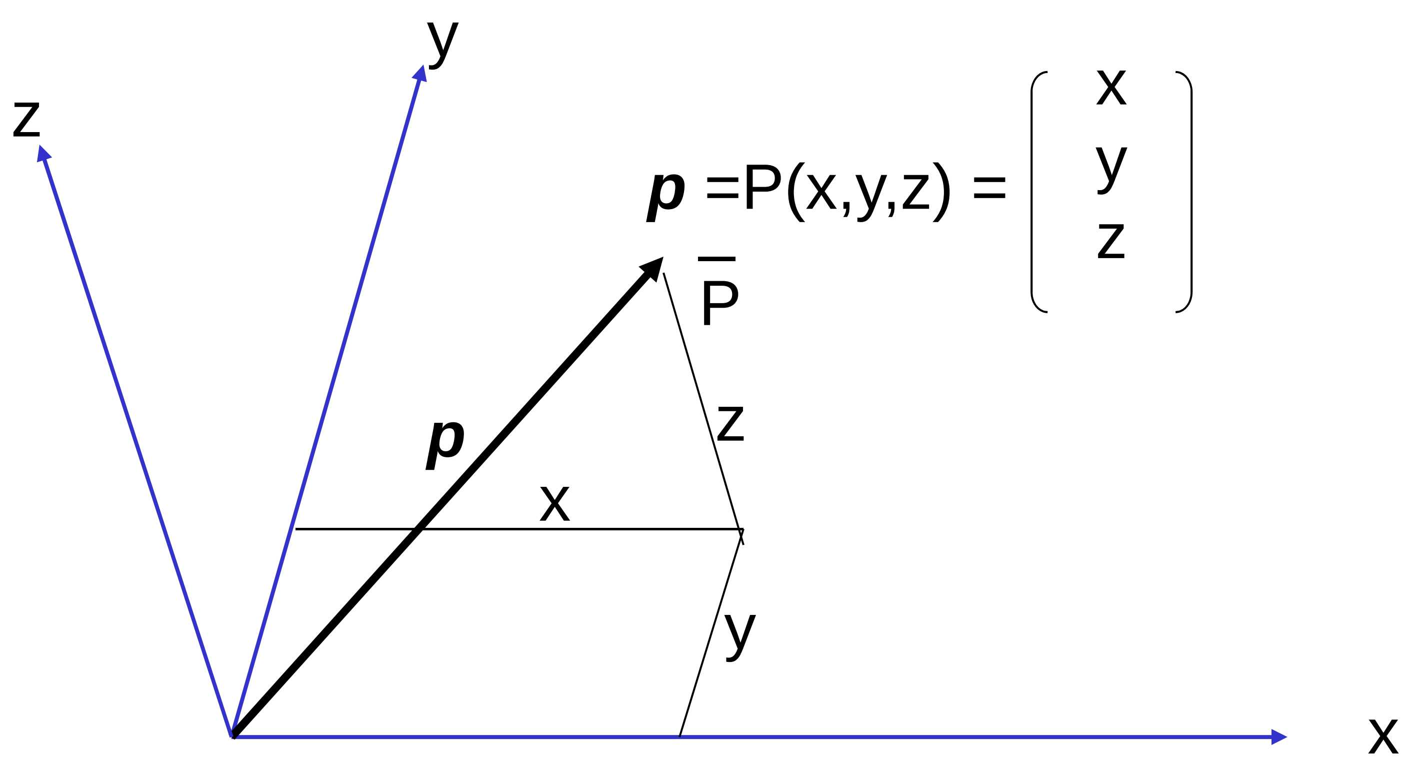
Basic Math – Vectors & Lines

Vector in 2D



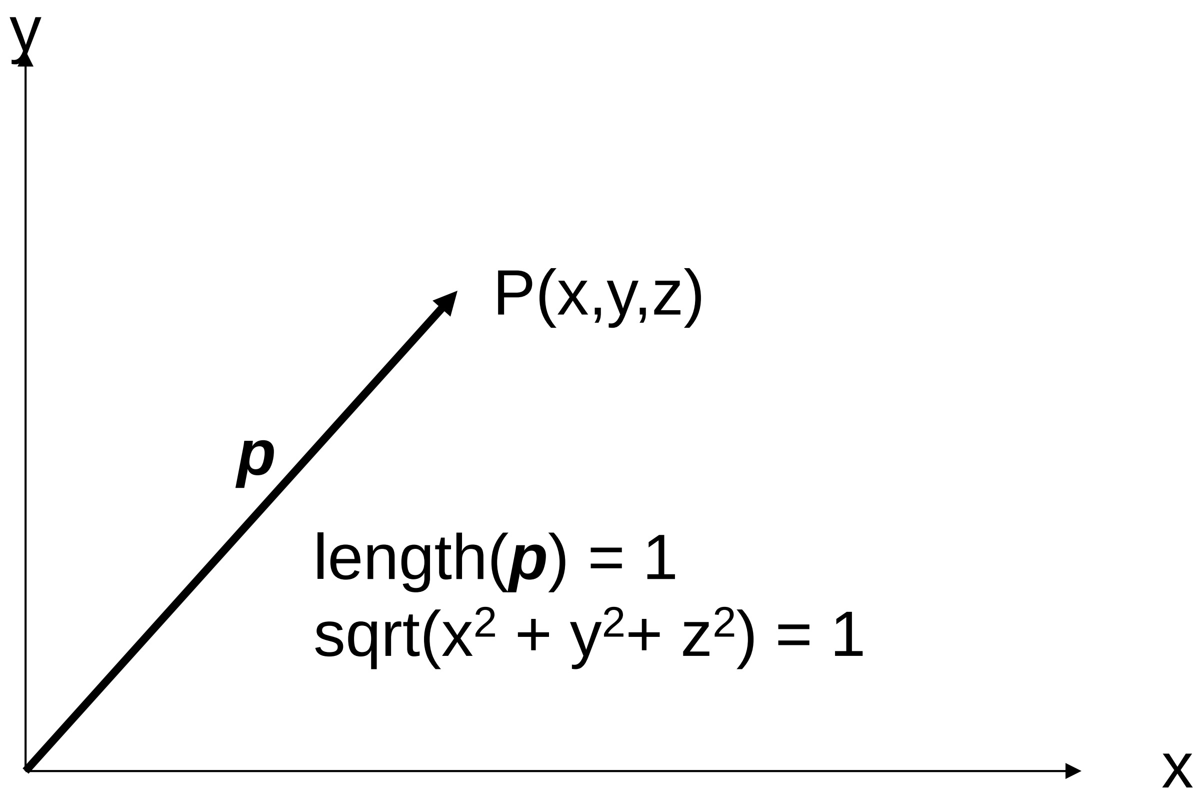
length ($\boldsymbol{p}$) = | $\boldsymbol{p}$ | = sqrt($x^2 + y^2$)

Vector in 3D



length ($\boldsymbol{p}$) = | $\boldsymbol{p}$ | = sqrt($x^2 + y^2 + z^2$)

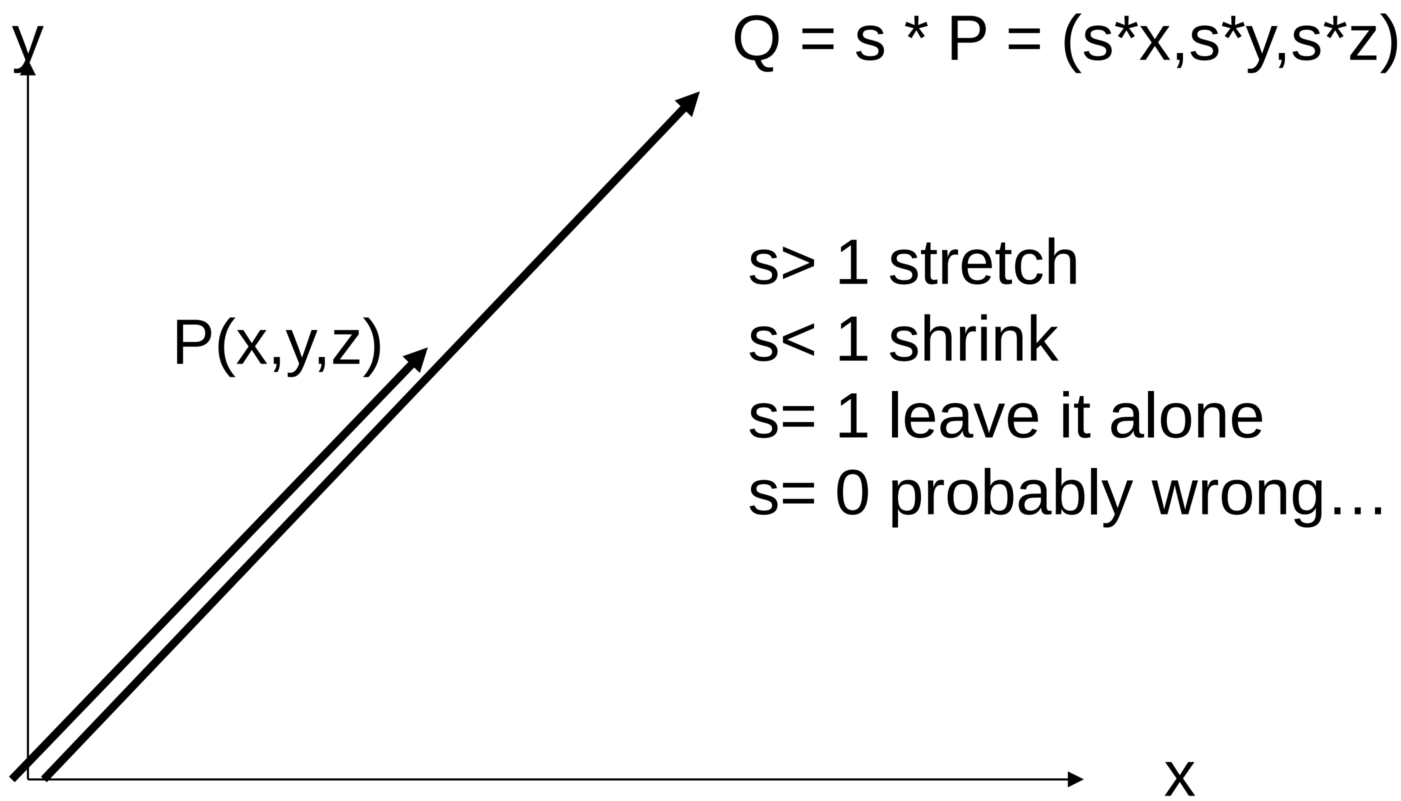
Unit vector



$$\text{length}(\boldsymbol{p}) = 1$$

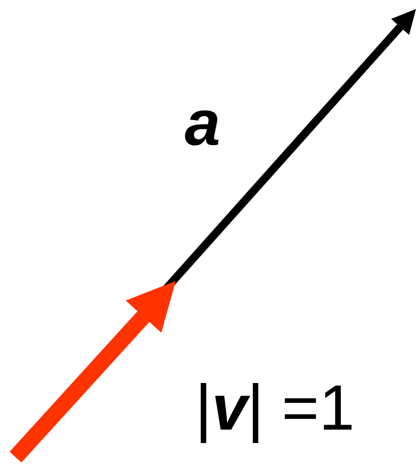
$$\text{sqrt}(x^2 + y^2 + z^2) = 1$$

Scaling up/down a vector



s> 1 stretch
s< 1 shrink
s= 1 leave it alone
s= 0 probably wrong…

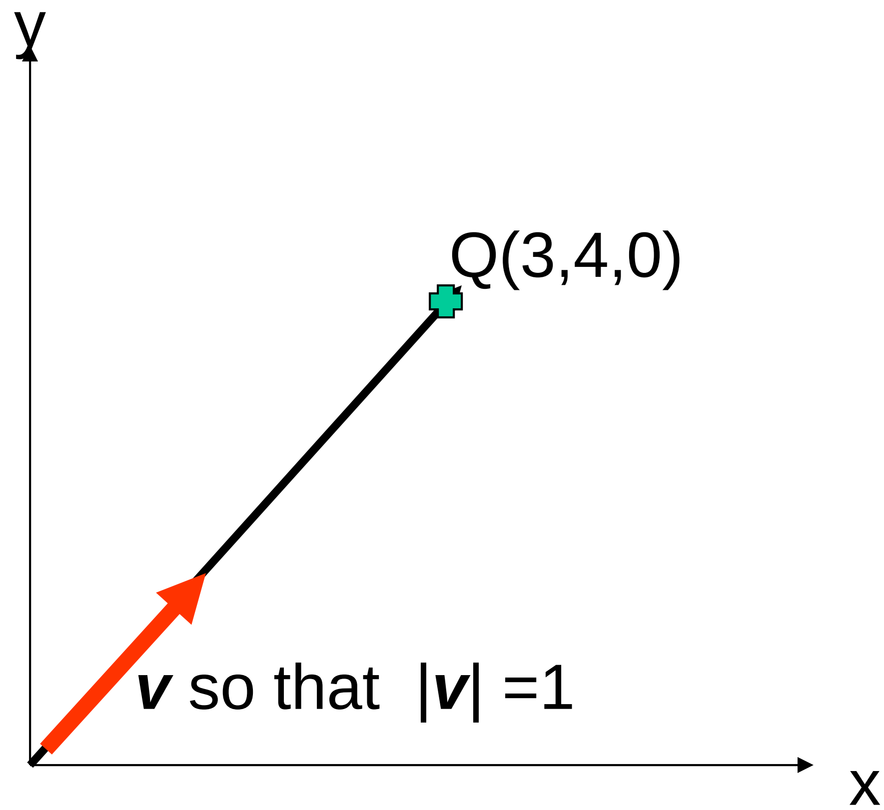
Create unit vector from a vector
a.k.a. “normalize” a vector



Scale down the vector by its own length:

v = ***a*** / |***a***| or ***v*** = ***a*** /length(***a***)

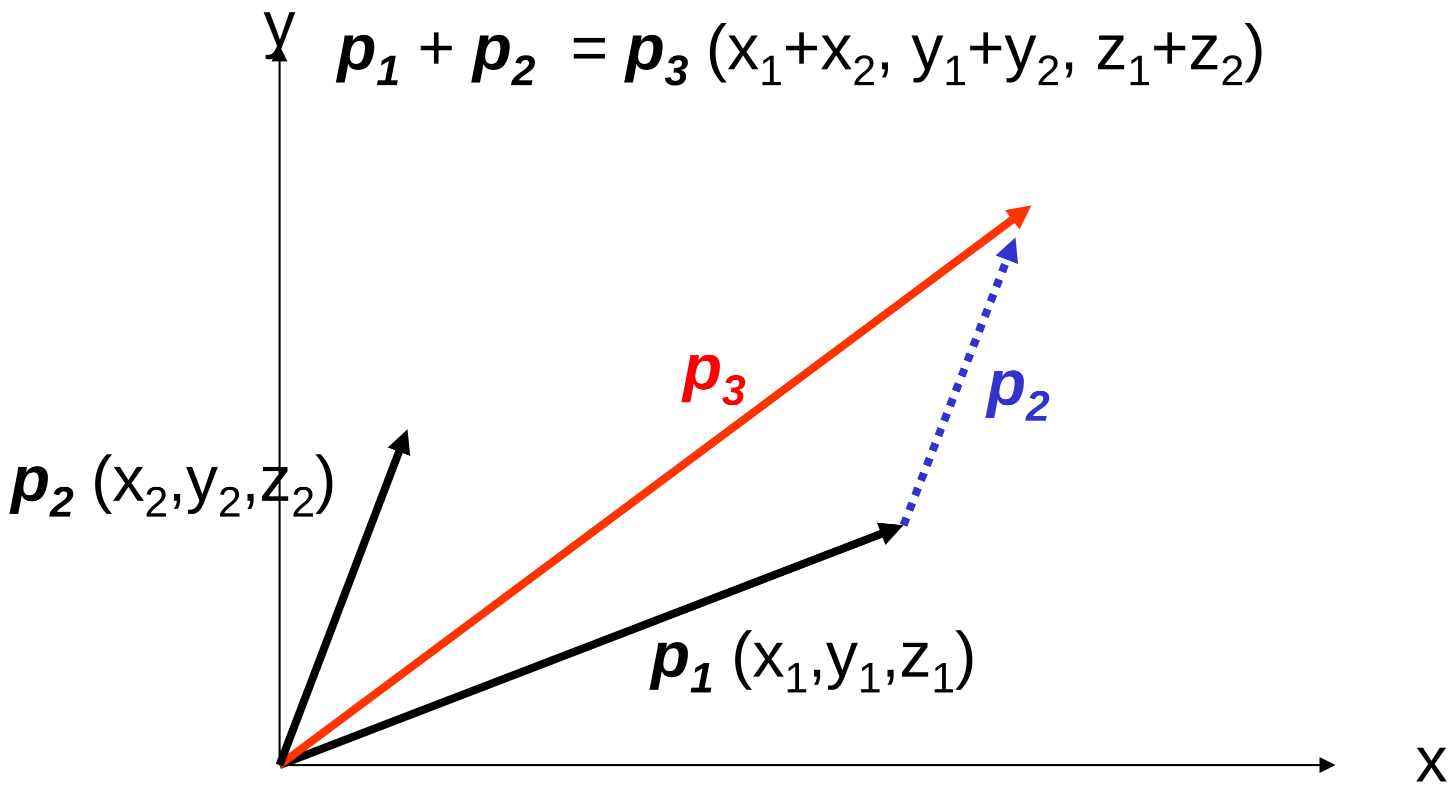
Create unit vector from a vector (Example)



length(Q) =sqrt(3*3 + 4*4+0*0) = sqrt(25) = 5

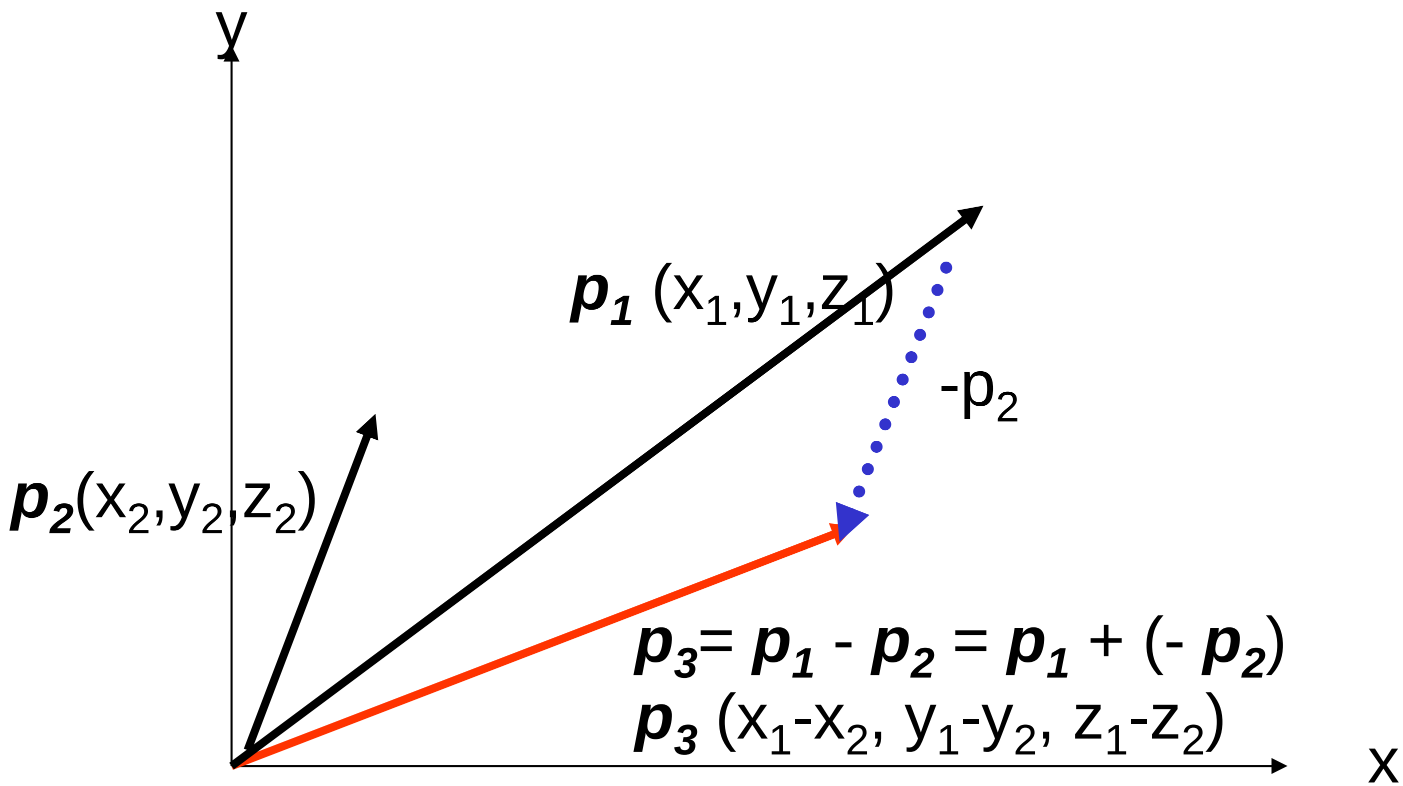
v = (3/5, 4/5, 0)

Sum of vectors



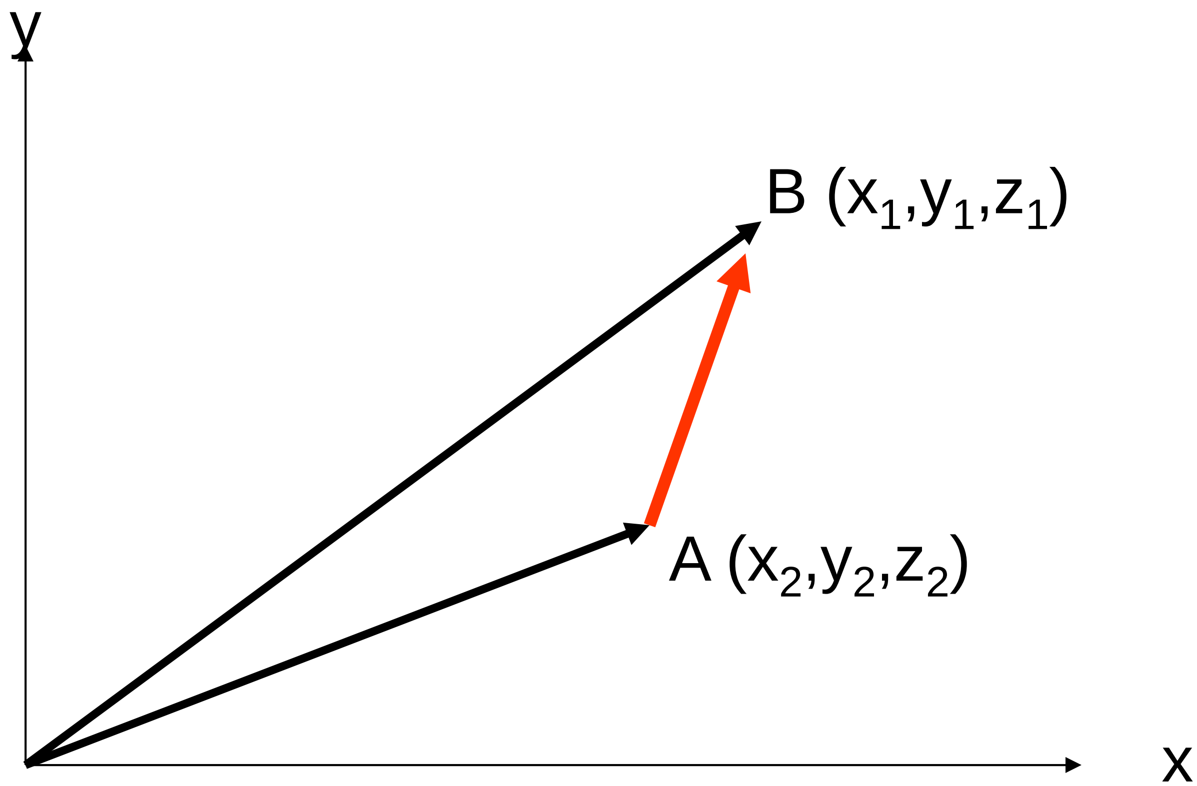
Catenate the vectors !

Subtraction of vectors



Reverse $\boldsymbol{p_2}$ and catenate to $\boldsymbol{p_1}$!

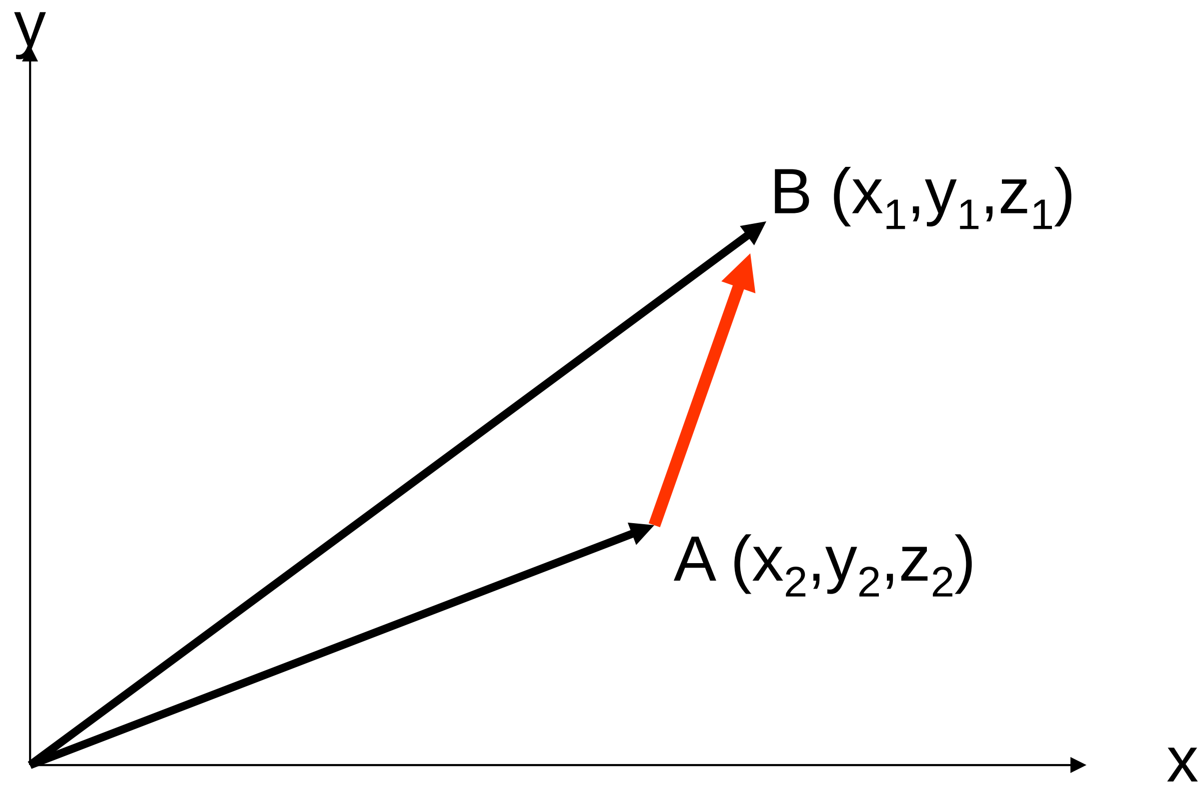
Vector from A to B



$AB = B - A = (x_1\text{-}x_2,\ y_1\text{-}y_2,\ z_1\text{-}z_2)$

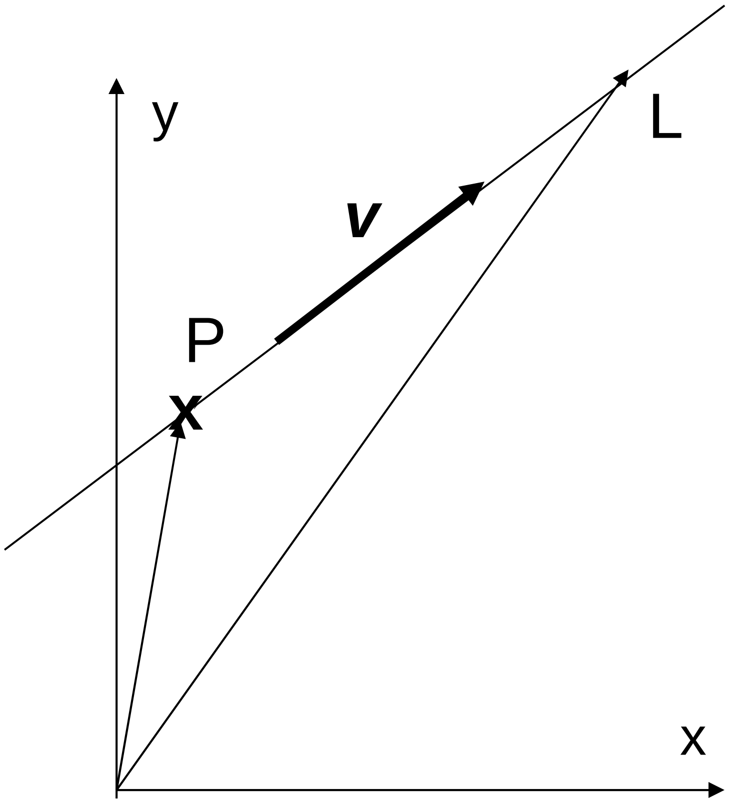
Subtract A from B

Distance between A to B



Distance = length(AB) = length(B – A) =
sqrt($(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2$)

Vector equation of a line



$L = P + t^* \boldsymbol{v}$

or broken to x,y,z components

$L_x = P_x + t^* v_x$

$L_y = P_y + t^* v_y$

$L_z = P_z + t^* v_z$

Infinite line:

t=(-inf,inf)

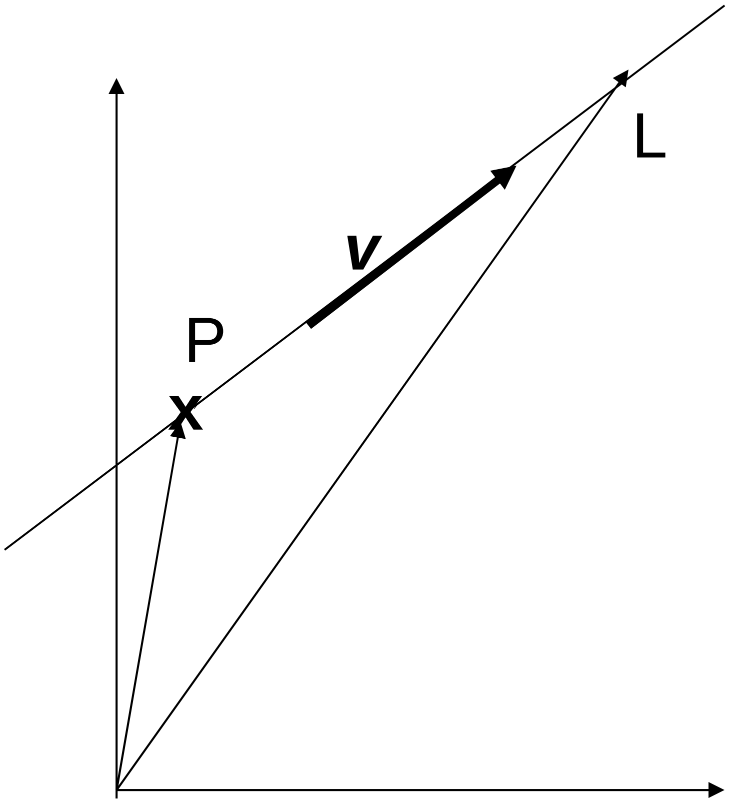
Ray:

t=(0,inf)

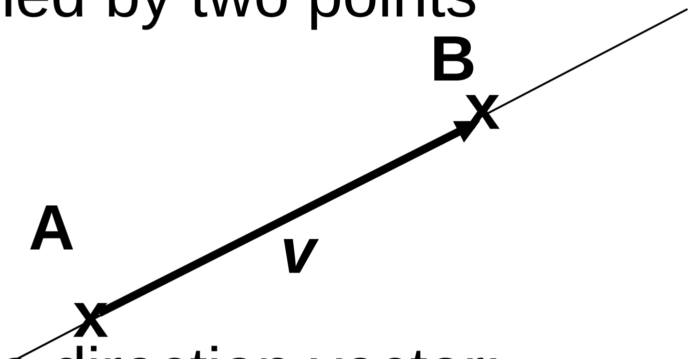
Segment:

$t=(t_{min}, t_{max})$

Create the direction vector of a line



Line defined by two points



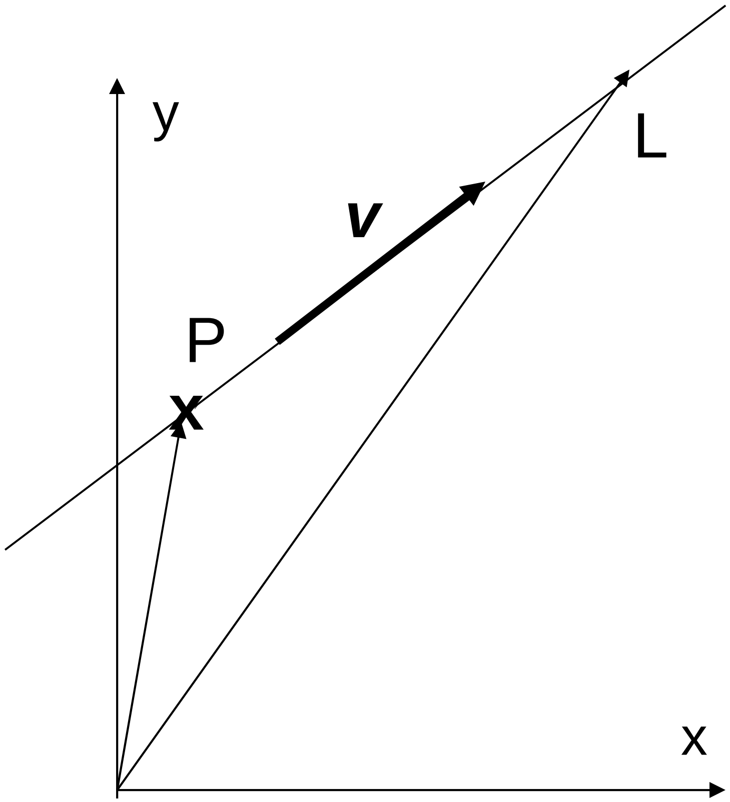
Obtain the direction vector:

AB = B – A   subtract

Optional:

v= AB / |AB| Normalize – if necessary

Vector equation of a line



$L = P + t * \boldsymbol{v}$

then

$L - P = t * \boldsymbol{v}$

then

$|L - P| = t|\boldsymbol{v}|$

Now if $|\boldsymbol{v}|=1$ (***v*** was normalized)

then

$|L - P| = t$

Cross product

Cross product (or vector product) of u and v is denoted as $\boldsymbol{q} = \boldsymbol{u} \times \boldsymbol{v}$, where ***u,v,q*** are 3D vectors denoted as $\boldsymbol{u}(x_1,y_1,z_1)$, $\boldsymbol{v}(x_2,y_2,z_2)$, $\boldsymbol{q}(x_3,y_3,z_3)$,

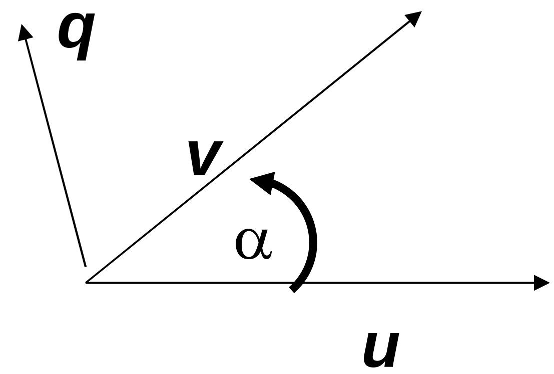
$|\boldsymbol{q}| = |\boldsymbol{u}| * |\underline{\mathbf{v}}| * \sin(\alpha)$

q is perpendicular to both ***u*** and ***v***, and

$x_3=y_1*z_2-y_2*z_1$

$y_3=x_2*z_1-x_1*z_2$

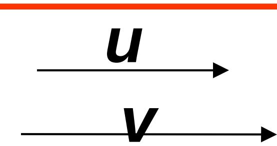
$z_3=x_1*y_2-x_2*y_1$



NOT COMMUTATIVE! ORDER MATTERS !

Cross product = 0 if and only if $\sin(\alpha)=0$

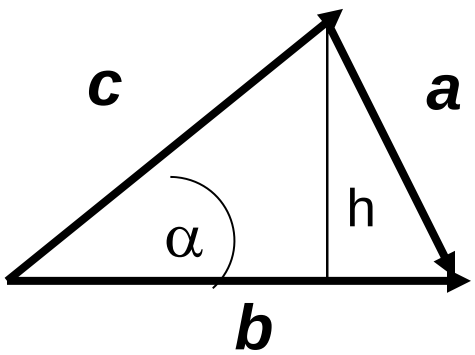
i.e. ***u*** and ***v*** are parallel



Area of a triangle

$A= ½ |\boldsymbol{b}|* h$ --- area of triangle

$h=|\boldsymbol{c}|*\sin(\alpha)$



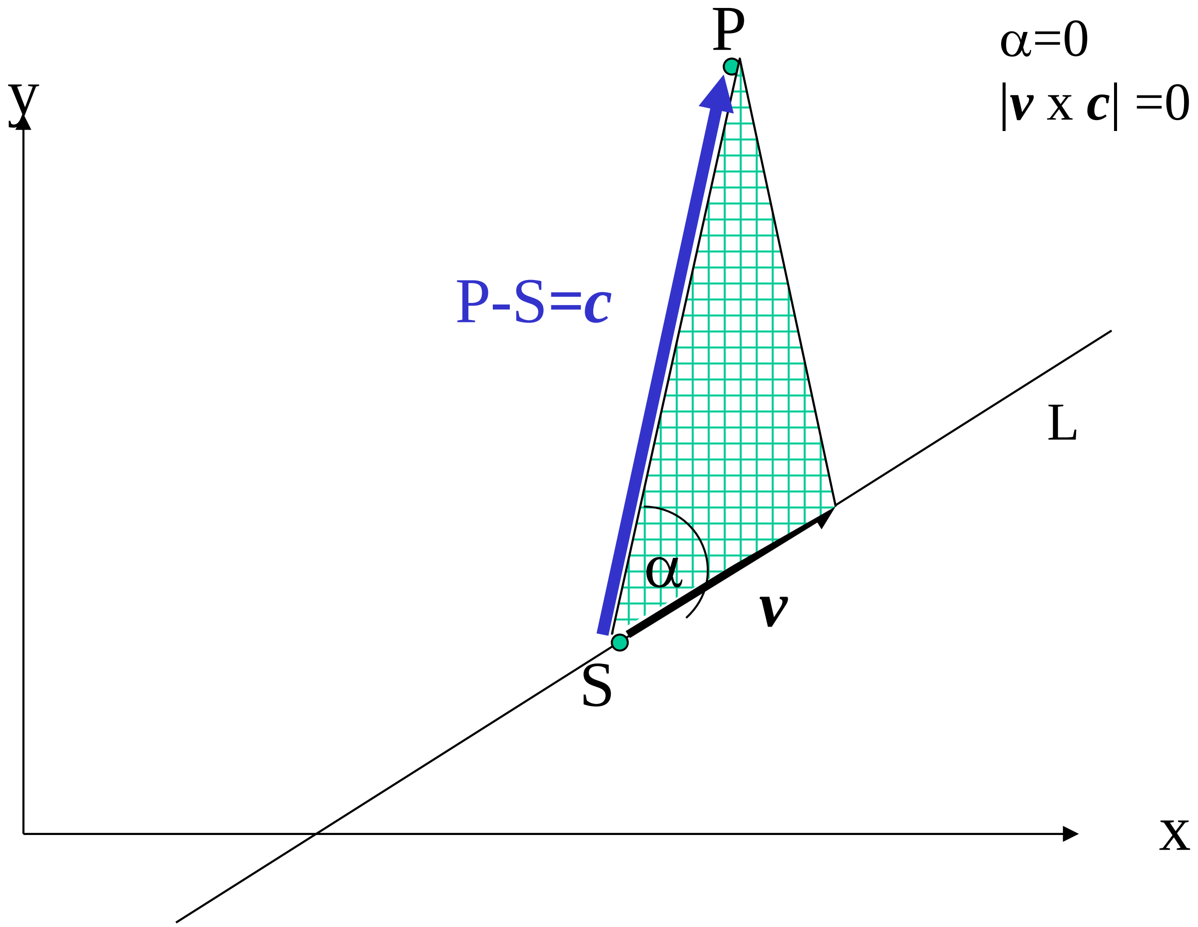
$A= ½ |\boldsymbol{b}|* |\boldsymbol{c}| *\sin(\alpha)$

$A= ½ | \boldsymbol{b} \times \boldsymbol{c} |$

For non-zero b and c, the area is 0 if and only if $\sin(\alpha) =0$

i.e. c and b are parallel, so $\alpha=0$

Is the point P on the line?

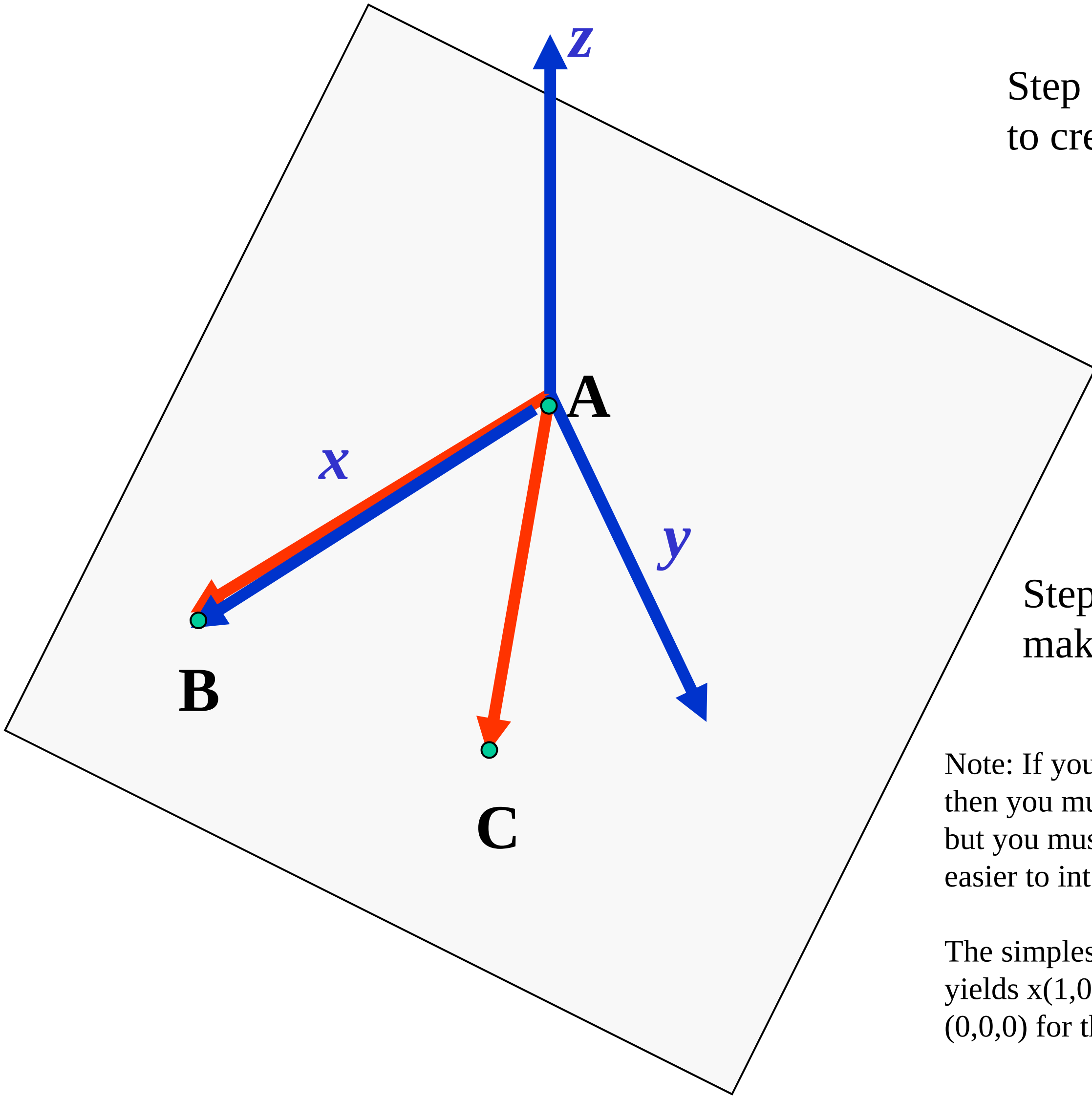


P is on the line i.o.i

$\alpha=0$

$|\boldsymbol{v} \times \boldsymbol{c}| = 0$

Ortho-normal vector base from 3 points, ABC



Step 1: Use cross products
to create three orthogonal vectors

x = **AB**

z = ***x*** x **AC**

y = ***z*** x ***x***

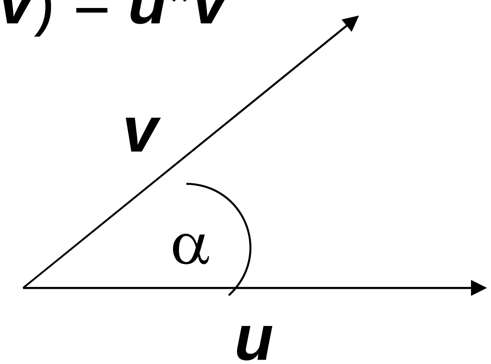
Step 2: Normalize x, y, and z to
make them unit length

Note: If you need to fix a coordinate system, to the ABC triangle, then you must also select a center. The center can be anywhere, but you must be consistent with the choice. Usually, it will be easier to interpret results if you fix the center in A.

The simplest test case is A(0,0,0), B(1,0,0) and C(0,1,0). This yields x(1,0,0), y(0,1,0) and z(0,0,1) for the three bases and (0,0,0) for the center.

Dot product

Dot product (or scalar product) = dot(***u***,***v***) = ***u********v***



u(x_1,y_1,z_1) ***v***(x_2,y_2,z_2)

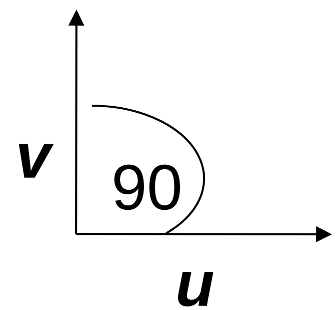
dot product = ***u*** * ***v*** = |***u***| * |***v***| * cos(α) $=(x_1x_2 + y_1y_2 + z_1z_2)$

Result is a scalar number, not a vector

It is commutative, so order does not matter

Dot product = 0 if and only if cos(α)=0,

i.e. ***u*** and ***v*** are perpendicular



Dot product and the length of a vector

Length = square root of the dot product with itself

$\boldsymbol{v}=(x,y,z)$

$\text{length}(\boldsymbol{v}) = \text{sqrt}(x^2 + y^2 + z^2) = \text{sqrt}(\ \text{dot}(\boldsymbol{v},\boldsymbol{v})\ )$

Some More Dot Product Facts

ab	= ba	commutative
(ab)c	!= a (bc)	not associative
(a + b)c	= ac + bc	distributive with addition

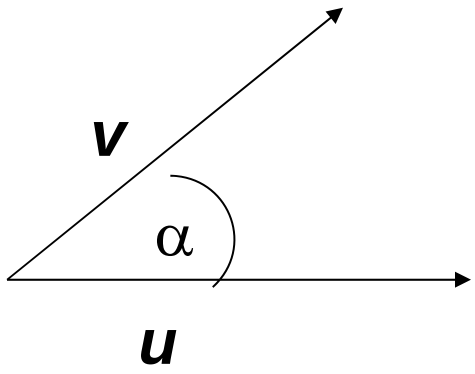
Angle between vectors

dot(***u***,***v***)=length(***u***) * length(***v***) * cos(α)
dot(***u***,***v***)= |***u***| * |***v***| * cos(α)

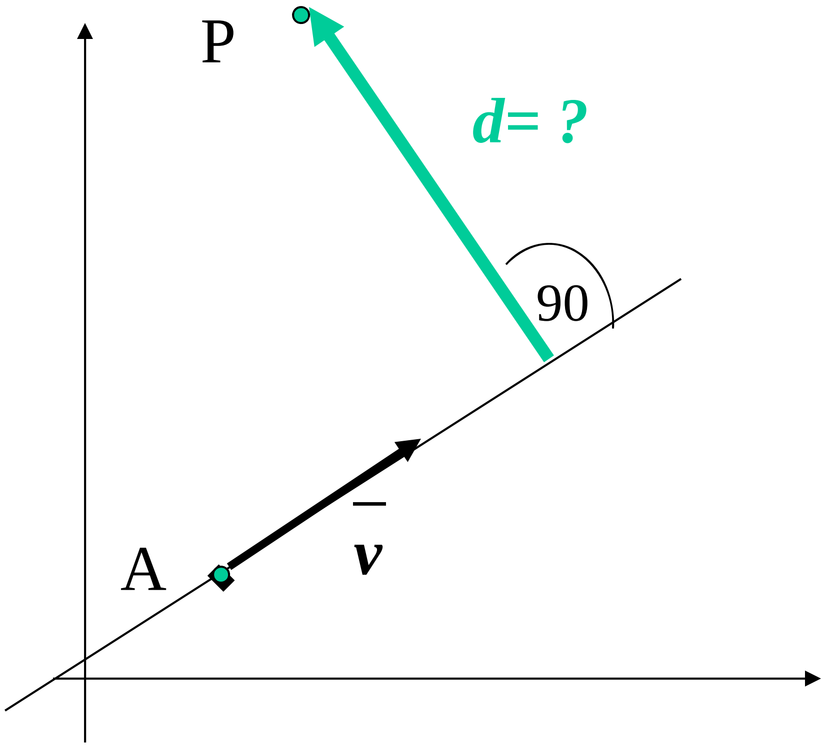
If ***u*** and ***v*** are unit vectors:

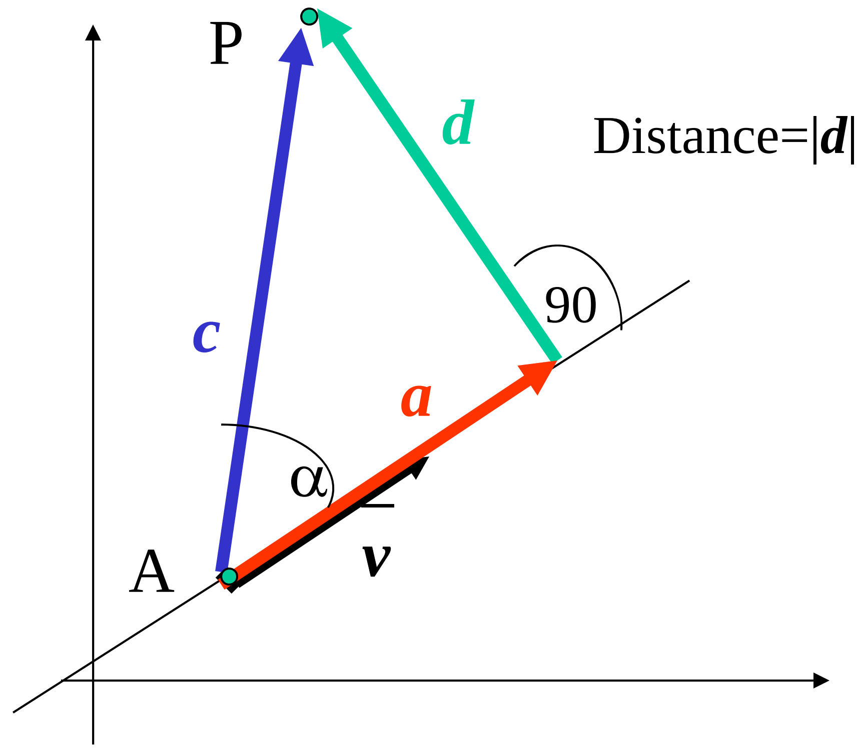
|***u***| =1 and |***v***| =1, so

dot(***u***,***v***) =cos(α)



Distance of a point from a line





$\boldsymbol{v}$ is a known unit vector, so length($\boldsymbol{v}$) =1

$\boldsymbol{c}$=P-A -- this is a known vector

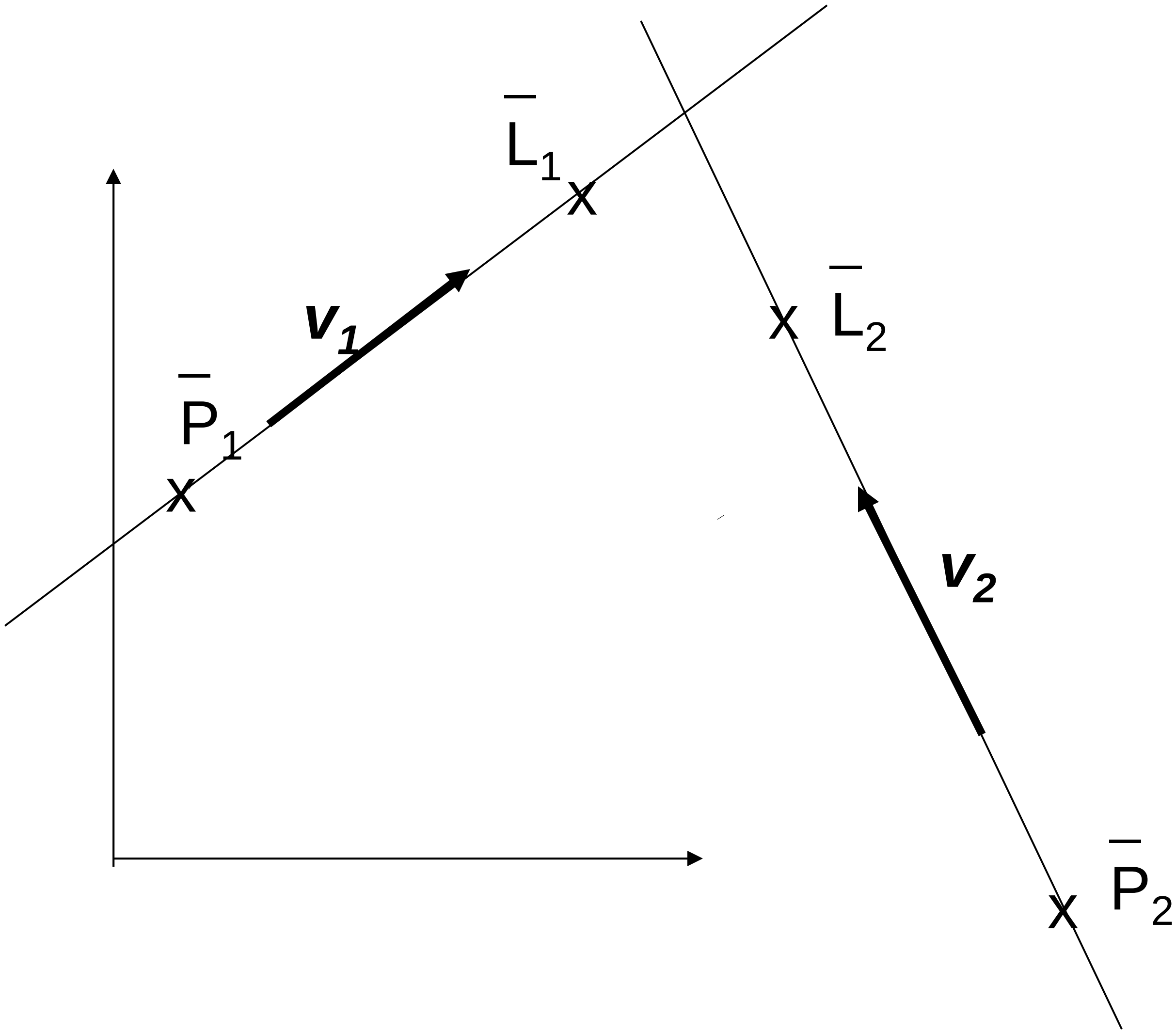
dot($\boldsymbol{v},\boldsymbol{c}$) = $\boldsymbol{vc} = |\boldsymbol{v}| * |\boldsymbol{c}| * \cos(\alpha) = |\boldsymbol{c}| * \cos(\alpha) = |\boldsymbol{a}|$

$\boldsymbol{a} = \boldsymbol{v} * |\boldsymbol{a}| = \boldsymbol{v}(\boldsymbol{vc})$

$\boldsymbol{d} = \boldsymbol{c} - \boldsymbol{a} = \boldsymbol{c} - \boldsymbol{v}(\boldsymbol{vc})$

dist = $|\mathbf{d}| = |\boldsymbol{c} - \boldsymbol{v}(\boldsymbol{vc})|$ or $\mathbf{d}^2 = \boldsymbol{c}^2 - (\boldsymbol{vc})^2$

Intersection of 2 lines?



$L_1 = P_1 + t_1 * \boldsymbol{v_1}$

$L_2 = P_2 + t_2 * \boldsymbol{v_2}$

$L_{1x} = P_{1x} + t_1 * v_{1x}$

$L_{1y} = P_{1y} + t_1 * v_{1y}$

$L_{1z} = P_{1z} + t_1 * v_{1z}$

$L_{2x} = P_{2x} + t_2 * v_{2x}$

$L_{2y} = P_{2y} + t_2 * v_{2y}$

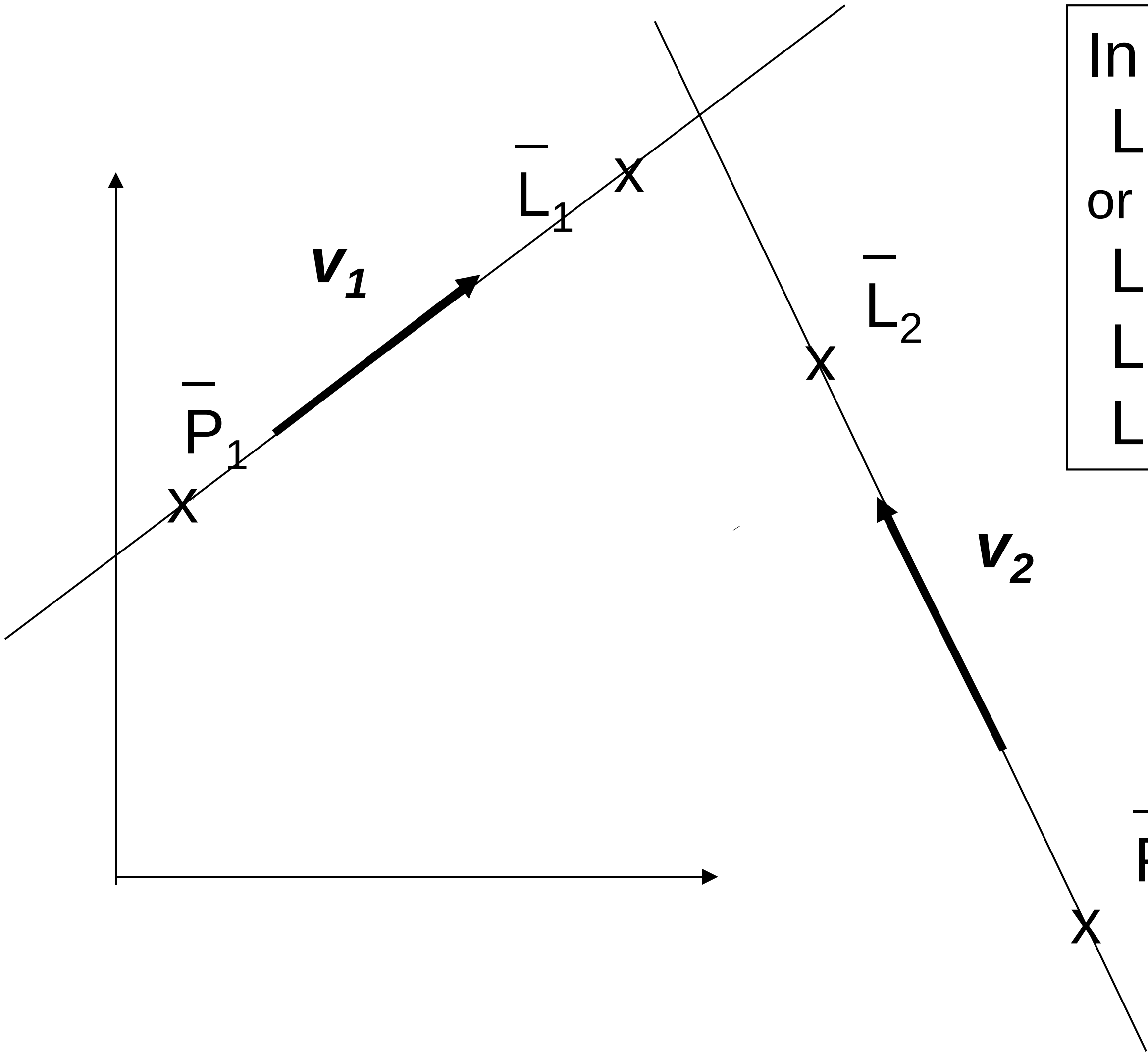
$L_{2z} = P_{2z} + t_2 * v_{2z}$

where

t=(-inf,inf)

u=(-inf,inf)

Intersection of 2 lines



In the intersection point :

$L_1 = L_2$

or by component

$L_{1x} = L_{2x}$

$L_{1y} = L_{2y}$

$L_{1z} = L_{2z}$

Intersection of 2 lines – IMPOSSIBLE

In the intersection point :

$$L_{1x} - L_{2x} = 0 = P_{1x} - P_{2x} + \mathbf{t_1} * v_{1x} - \mathbf{t_2} * v_{2x}$$
$$L_{1y} - L_{2y} = 0 = P_{1y} - P_{2y} + \mathbf{t_1} * v_{1y} - \mathbf{t_2} * v_{2y}$$
$$L_{1z} - L_{2z} = 0 = P_{1z} - P_{2z} + \mathbf{t_1} * v_{1z} - \mathbf{t_2} * v_{2z}$$

where

t=(-inf,inf)

u=(-inf,inf)

Trouble: 3 eqs, 2 unknowns → no guaranteed solution.

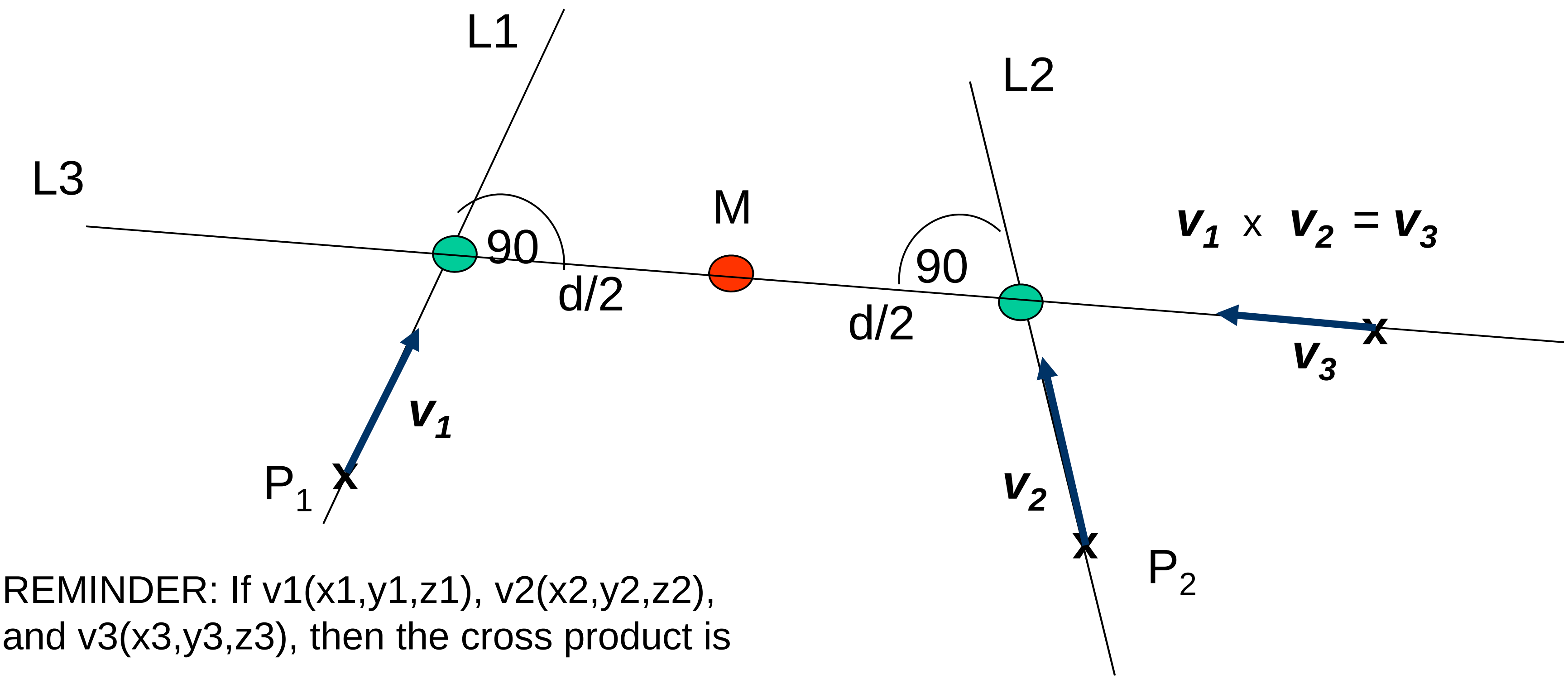
The lines might just intersect, but they do not have to.

When they intersect, one of the three eqs cancels out. In other words: a linear combination of any two gives the third one. (We have to be extremely lucky for this to happen.) Generally, two lines avoid each other.

We approximate: find the shortest distance between the two lines and find the point in midway.

Approximate Intersection of 2 lines

Intuition: Find the line that is perpendicular to both lines



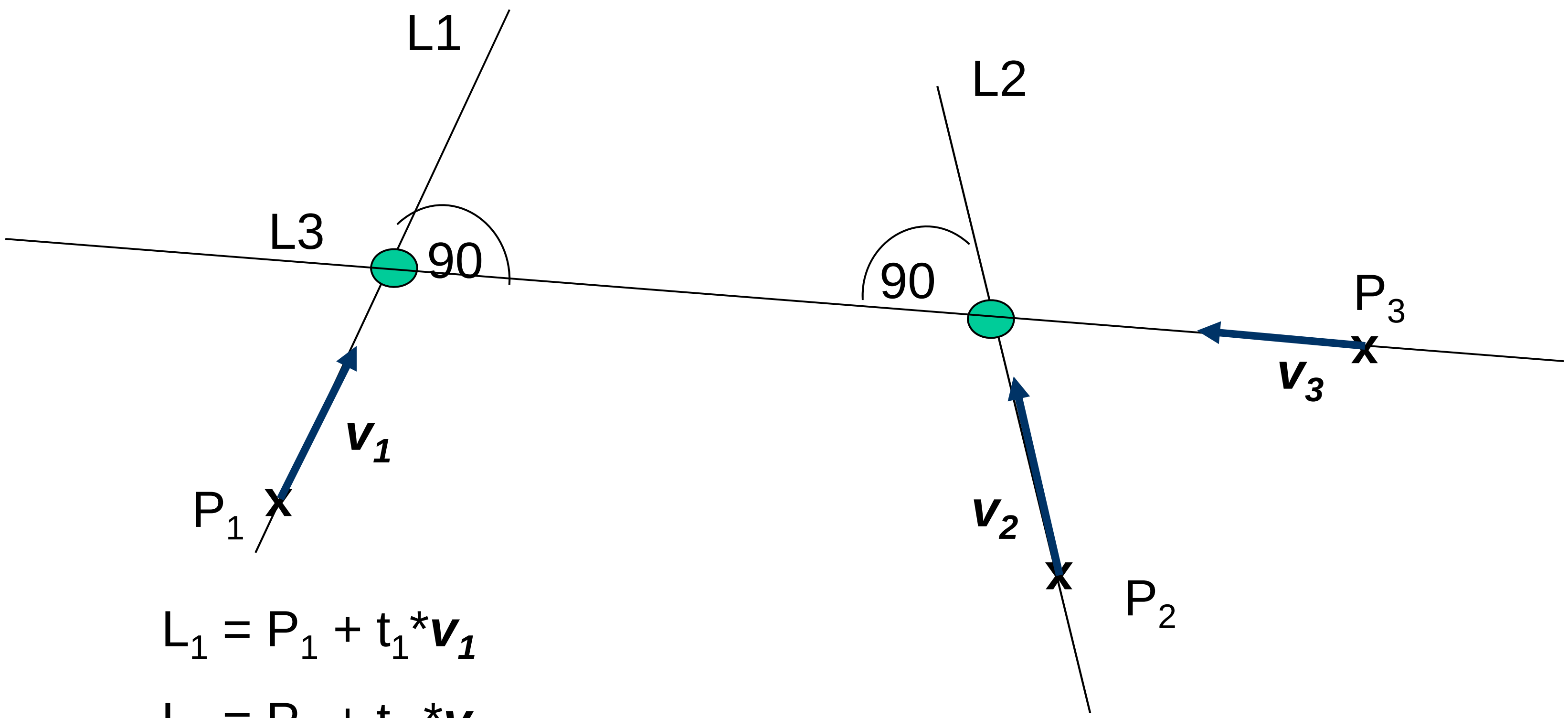
REMINDER: If v1(x1,y1,z1), v2(x2,y2,z2), and v3(x3,y3,z3), then the cross product is

x3=y1*z2-y2*z1

y3=x2*z1-x1*z2

z3=x1*y2-x2*y1

Write up the equation of each line

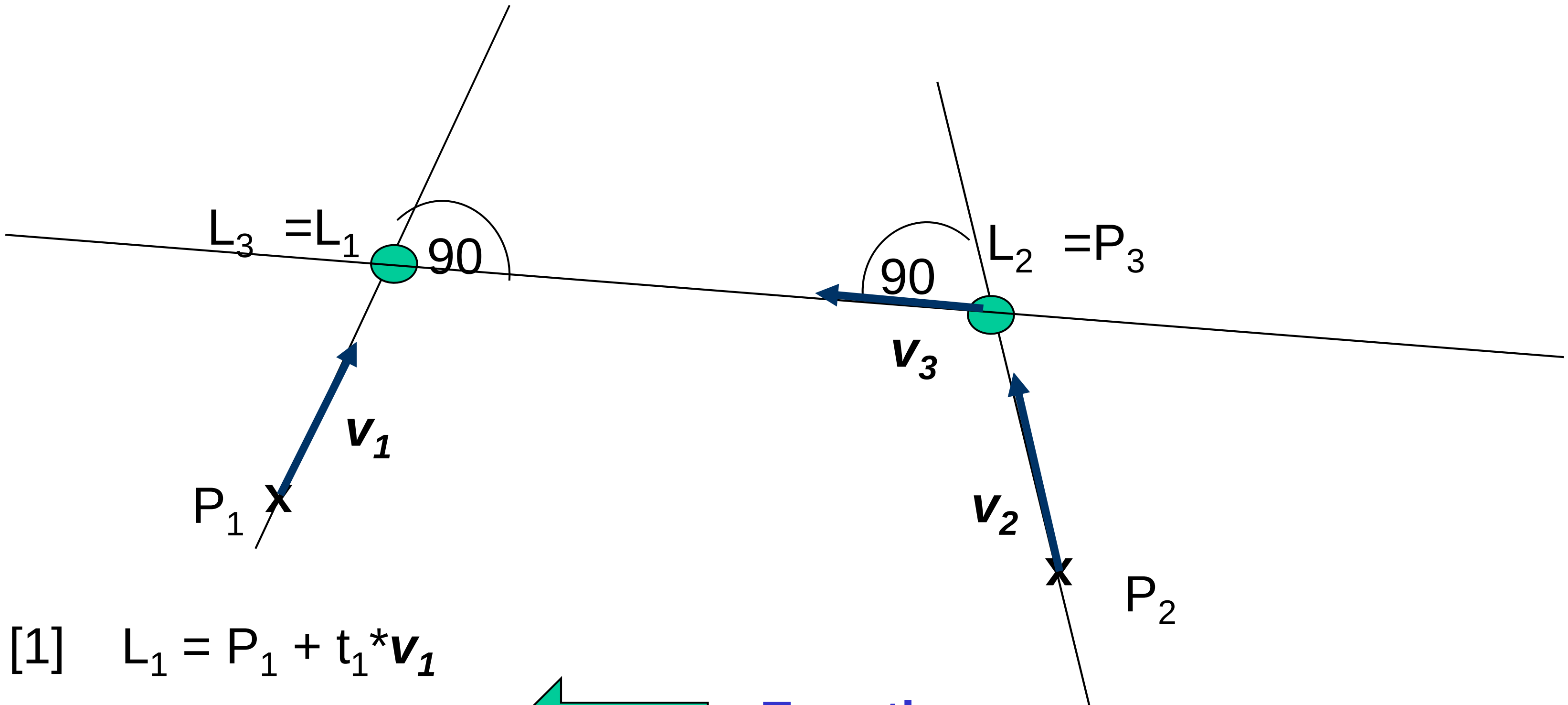


$L_1 = P_1 + t_1 * \boldsymbol{v_1}$

$L_2 = P_2 + t_2 * \boldsymbol{v_2}$

$L_3 = P_3 + t_3 * \boldsymbol{v_3}$

Derive conditions, make an equation system



[1] $L_1 = P_1 + t_1 * \boldsymbol{v_1}$

[2] $L_2 = P_2 + t_2 * \boldsymbol{v_2}$

Equations

[3] $L_3 = P_3 + t_3 * \boldsymbol{v_3}$, where

$P_3 = L_2$

$L_3 = L_1$

$\boldsymbol{v_1} \times \boldsymbol{v_2} = \boldsymbol{v_3}$

Conditions

Solve the vector equation system

[1] $L_1 = P_1 + t_1 * \boldsymbol{v_1}$

[2] $L_2 = P_2 + t_2 * \boldsymbol{v_2}$

[3] $L_3 = P_3 + t_3 * \boldsymbol{v_3}$

where

$P_3 = L_2$ -- replace P_3 in [3]

$L_3 = L_1$ -- replace L_3 in [3]

[1] $L_1 = P_1 + t_1 * \boldsymbol{v_1}$ *-- plug [1] in [3]*

[2] $L_2 = P_2 + t_2 * \boldsymbol{v_2}$ *-- plug [1] in [3]*

[3] $L_1 = L_2 + t_3 * \boldsymbol{v_3}$

$P_1 + t_1 * \boldsymbol{v_1} = P_2 + t_2 * \boldsymbol{v_2} + t_3 * \boldsymbol{v_3}$ *– arrange*

$P_1 - P_2 = -t_1 * \boldsymbol{v_1} + t_2 * \boldsymbol{v_2} + t_3 * \boldsymbol{v_3}$ *– break it up*

$P_{1x} - P_{2x} = -t_1 * v_{1x} + t_2 * v_{2x} + t_3 * v_{3x}$

$P_{1y} - P_{2y} = -t_1 * v_{1y} + t_2 * v_{2y} + t_3 * v_{3y}$

$P_{1z} - P_{2z} = -t_1 * v_{1z} + t_2 * v_{2z} + t_3 * v_{3z}$

- 3 linear equations
- 3 unknowns
- SOLVABLE

Solve the linear equations

Use any of the three methods:

1. Gaussian elimination
2. Substitution
3. Matrix inversion

$$\begin{pmatrix} P_{1x} - P_{2x} \\ P_{1y} - P_{2y} \\ P_{1z} - P_{2z} \end{pmatrix} = \begin{pmatrix} -v_{1x} & v_{2x} & v_{3x} \\ -v_{1y} & v_{2y} & v_{3y} \\ -v_{1z} & v_{2z} & v_{3z} \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$$

$$\begin{pmatrix} -v_{1x} & v_{2x} & v_{3x} \\ -v_{1y} & v_{2y} & v_{3y} \\ -v_{1z} & v_{2z} & v_{3z} \end{pmatrix}^{-1} \begin{pmatrix} P_{1x} - P_{2x} \\ P_{1y} - P_{2y} \\ P_{1z} - P_{2z} \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$$

Plug t_1 and t_2 back into L_1 and L_2 line equations

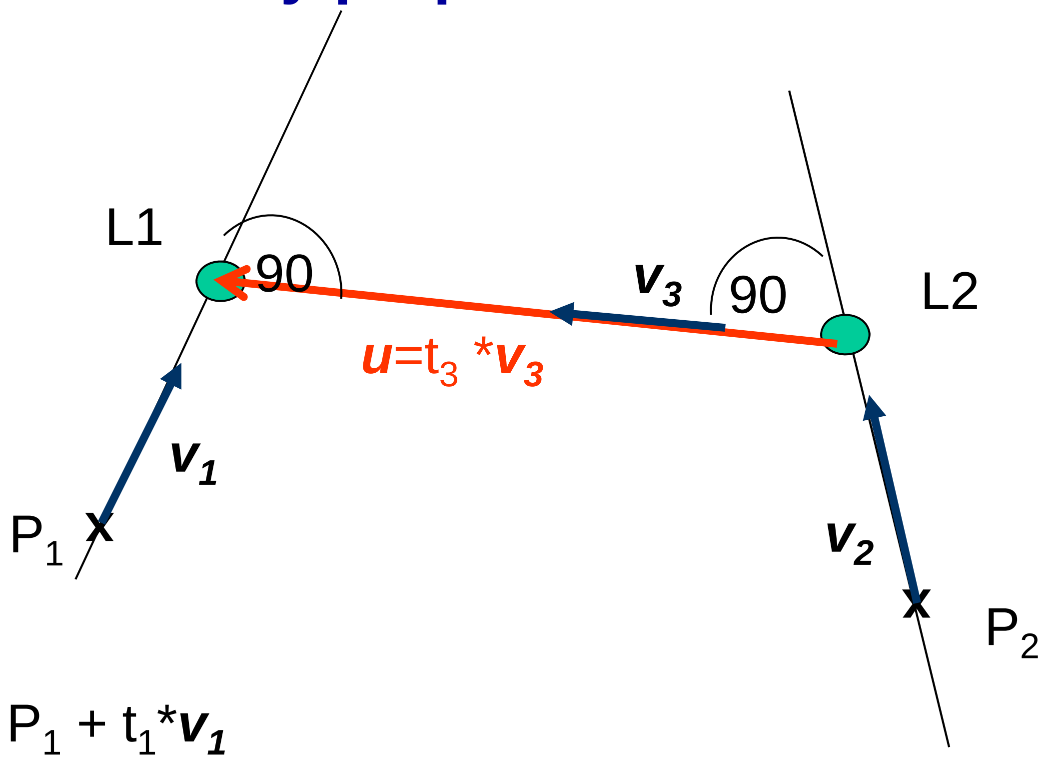
$$L_1 = P_1 + t_1 * \boldsymbol{v_1}$$

$$L_2 = P_2 + t_2 * \boldsymbol{v_2}$$

$$M = (L_1 + L_2)/2$$

$$d = \text{length}(L_1 - L_2)$$

Find the mutually perpendicular vector between L1 & L2



$L_1 = P_1 + t_1 * \boldsymbol{v_1}$

$L_2 = P_2 + t_2 * \boldsymbol{v_2}$

$\boldsymbol{v_3} = \boldsymbol{v_1} \times \boldsymbol{v_2} / |\boldsymbol{v_1} \times \boldsymbol{v_2}|$

***v_3* is normalized!!!**

We look for the vector ***u*** that is perpendicular to both L1 and L2. This vector is expressed as $\boldsymbol{u} = t_3 * \boldsymbol{v_3}$. The length of it is t_3 and its direction is determined by the $\boldsymbol{v_3}$ unit direction vector. $\boldsymbol{v_3}$ is the cross product of $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$, being perpendicular to both.

Find the mutually perpendicular vector between L1 & L2

$$L_1 - L_2 = t_3 * \boldsymbol{v_3}$$

$$P_1 - P_2 + t_1 * \boldsymbol{v_1} - t_2 * \boldsymbol{v_2} = t_3 * \boldsymbol{v_3}$$

$$P_1 - P_2 = -t_1 * \boldsymbol{v_1} + t_2 * \boldsymbol{v_2} + t_3 * \boldsymbol{v_3}$$

NICE TRICK !!!

– dot produce by $\boldsymbol{v_3}$

$$(P_1 - P_2)\, \boldsymbol{v_3} = -t_1 * \underbrace{\boldsymbol{v_1 v_3}}_{0} + t_2 * \underbrace{\boldsymbol{v_2 v3}}_{0} + t_3 * \underbrace{\boldsymbol{v_3 v_3}}_{1}$$

$$\mathbf{(P_1 - P_2)\, \boldsymbol{v_3} = t_3}$$

☺

Use the same "dot product trick" for t_1 and t_2

$$P_1 - P_2 = -t_1 * \boldsymbol{v_1} + t_2 * \boldsymbol{v_2} + t_3 * \boldsymbol{v_3}$$

---

(1) $$(P_1 - P_2)\,\boldsymbol{v_1} = -t_1 * \underbrace{\boldsymbol{v_1}\,\boldsymbol{v_1}}_{1} + t_2 * \boldsymbol{v_1}\,\boldsymbol{v_2} + t_3 * \underbrace{\boldsymbol{v_1}\,\boldsymbol{v_3}}_{0}$$ *if dot produce by* $\boldsymbol{v_1}$

(2) $$(P_1 - P_2)\,\boldsymbol{v_2} = -t_1 * \boldsymbol{v_2}\,\boldsymbol{v_1} + t_2 * \underbrace{\boldsymbol{v_2}\,\boldsymbol{v_2}}_{1} + t_3 * \underbrace{\boldsymbol{v_2}\,\boldsymbol{v_3}}_{0}$$ *if dot produce by* $\boldsymbol{v_2}$

We recognize three dot products of vectors:

$(P_1 - P_2)\, v_1$ and $(P_1 - P_2)\, v_1$ and $v_1 v_2$

REMEMBER: dot product is a scalar number

Let a_1, a_2 and d denote those scalars..

(1) $$\underbrace{(P_1 - P_2)\,\boldsymbol{v_1}}_{a_1} = -t_1 * \underbrace{\boldsymbol{v_1}\,\boldsymbol{v_1}}_{1} + t_2 * \underbrace{\boldsymbol{v_1}\,\boldsymbol{v_2}}_{d} + t_3 * \underbrace{\boldsymbol{v_1}\,\boldsymbol{v_3}}_{0}$$

(2) $$\underbrace{(P_1 - P_2)\,\boldsymbol{v_2}}_{a_2} = -t_1 * \underbrace{\boldsymbol{v_1}\,\boldsymbol{v_2}}_{d} + t_2 * \underbrace{\boldsymbol{v_2}\,\boldsymbol{v_2}}_{1} + t_3 * \underbrace{\boldsymbol{v_2}\,\boldsymbol{v_3}}_{0}$$

(1) $a_1 = -t_1 + t_2 d$

(2) $a_2 = -t_1 d + t_2$

This solves easily for t1 and t2.

$t_1 = (a_2 d - a_1)(1-d^2)$

$t_2 = -(a_1 d - a_2)(1-d^2)$

☺

$\mathbf{t_3 = (P_1 - P_2)\,\boldsymbol{v_3}}$

Plug t_1 and t_2 back into L_1 and L_2 line equations

$L_1 = P_1 + t_1 * \boldsymbol{v_1}$

$L_2 = P_2 + t_2 * \boldsymbol{v_2}$

The mid point is $M = (L_1 + L_2)/2$

The distance between L_1 and L_2 is t_3