Basic Math – Matrices & Transformations (revised)

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Matrices

An ordered table of numbers (or sub-tables)

columns

$$\underbrace{\mathbf{A}}_{3\times 3} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix}$$

Rows

$$\text{Product} \quad : \mathbf{C} = \mathbf{BA} \neq \mathbf{AB} \,,$$

$$\mathrm{C} = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} a_{00} * b_{00} + a_{01} * b_{10} & a_{00} * b_{01} + a_{01} * b_{11} \\ a_{10} * b_{00} + a_{11} * b_{10} & a_{10} * b_{01} + a_{11} * b_{11} \end{bmatrix}$$

$$\text{Identity} \quad : \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Matrix Addition (+)

Definition:

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} + \begin{bmatrix} x & u \\ y & v \end{bmatrix} = \begin{bmatrix} a+x & c+u \\ b+y & d+v \end{bmatrix}$$

Example:

$$\begin{bmatrix} 1 & 5 \\ 2 & 6 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 1 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$$

Properties:

1. Closed — result is always a matrix
2. Commutative — A+B = B+A
3. Associative — (A+B)+C = A+(B+C)
4. Identity — A+0 = A (null matrix)
5. Inverse — $A+A^{-}=0$ (negated matrix)

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Matrix Multiplication (*)

Definition:

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} * \begin{bmatrix} x & u \\ y & v \end{bmatrix} = \begin{bmatrix} ax+cy & au+cv \\ bx+dy & bu+dv \end{bmatrix}$$

Example:

$$\begin{bmatrix} 1 & 5 \\ 2 & 6 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 1 & -7 \end{bmatrix} = \begin{bmatrix} 5 & -37 \\ 6 & -46 \end{bmatrix}$$

Properties:

1.	Closed	result is always a matrix
2.	NOT Commutative	A*B != B*A
3.	Associative	(A*B)*C = A*(B*C)
4.	Identity	A*I = A (identity matrix)
5.	Inverse	$A*A^{-}=I$ (difficult to obtain!)

Distributive from either side

(A+B)C = AC + BC

C(A+B) = CA + CB

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Vector is a special matrix

Examples:

v=[x,y,z] row vector, 1x3 matrix

$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ column vector, 2x1 matrix

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Vector addition: just like adding matrices

Column vector

$$\begin{bmatrix} a & \cancel{c} \\ b & \cancel{d} \end{bmatrix} + \begin{bmatrix} x & \cancel{u} \\ y & \cancel{v} \end{bmatrix} = \begin{bmatrix} a+x & \cancel{c+u} \\ b+y & \cancel{d+v} \end{bmatrix}$$

Row vector

$$\begin{bmatrix} a & c \\ \cancel{b} & \cancel{d} \end{bmatrix} + \begin{bmatrix} x & u \\ \cancel{y} & \cancel{v} \end{bmatrix} = \begin{bmatrix} a+x & c+u \\ \cancel{b+y} & \cancel{d+v} \end{bmatrix}$$

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Matrix*Vector Multiplication:
just like multiplying matrices

$$\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

Column vector on the right

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{M}\mathbf{v}_1$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} ax_1 + by_1 \\ cx_1 + dy_1 \end{bmatrix}$$

Result is a new column vector

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Matrix*Vector Multiplication(example)

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} 5 & 1 \\ 2 & 3 \end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{M}\mathbf{v}_1$$

$$= \begin{bmatrix} 5 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5*1+1*2 \\ 2*1+3*2 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

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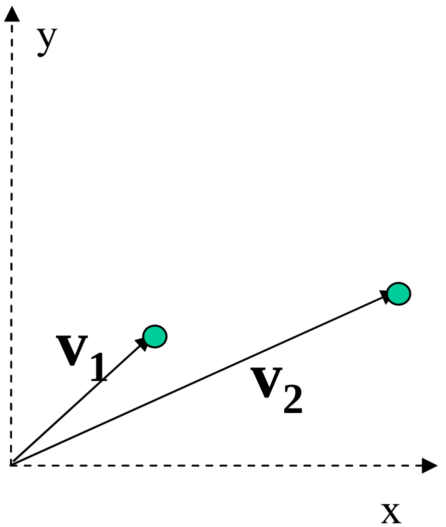
Matrices as transformations

Matrix-vector multiplication creates a new vector.
New vector means new location in space.
If all points of an object is multiplied by a matrix,
then the whole object assumes a new position (and may be new shape and size, too). This is called transformation.

$$\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{M}\mathbf{v}_1$$



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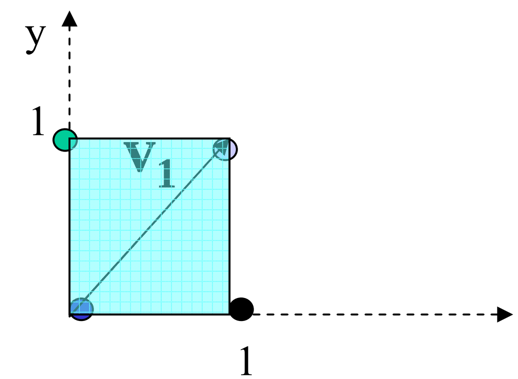
Matrices as transformations (examples)

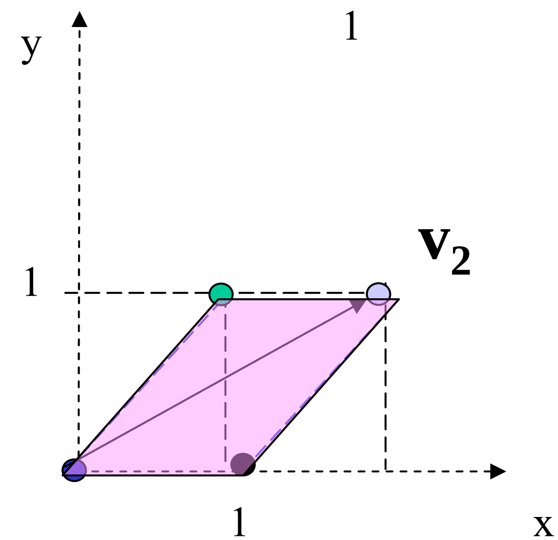
Transform 4 points [0,0], [1,0],[1,1],[0,1] using the matrix M.

$$\mathbf{M} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{v}_2 = \mathbf{M}\mathbf{v}_1$$

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$





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Scaling

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{S}\mathbf{v}_1$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5*1+0*2 \\ 0*1+3*2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \qquad \mathbf{S}^{-1} = \begin{bmatrix} 1/s_x & 0 \\ 0 & 1/s_y \end{bmatrix} \qquad \mathbf{S} * \mathbf{S}^{-1} = \mathbf{I}$$

$$\boxed{\mathbf{S} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix} \quad \text{3D scaling matrix}}$$

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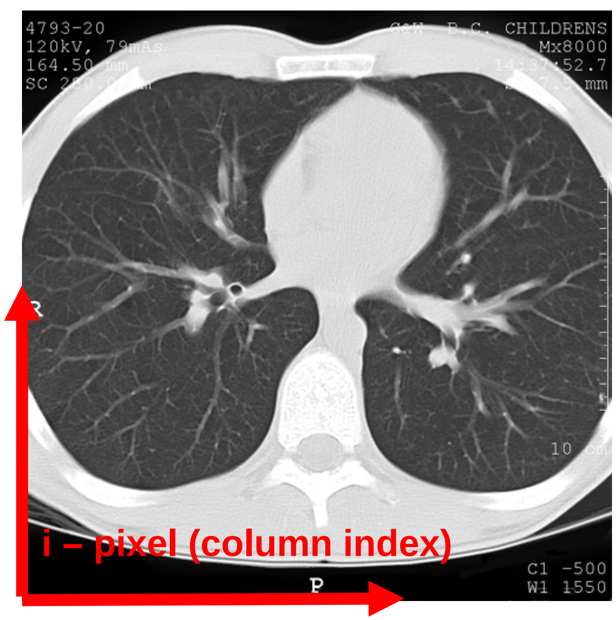
Example: CT guidance



NY number of pixels (typically 512)

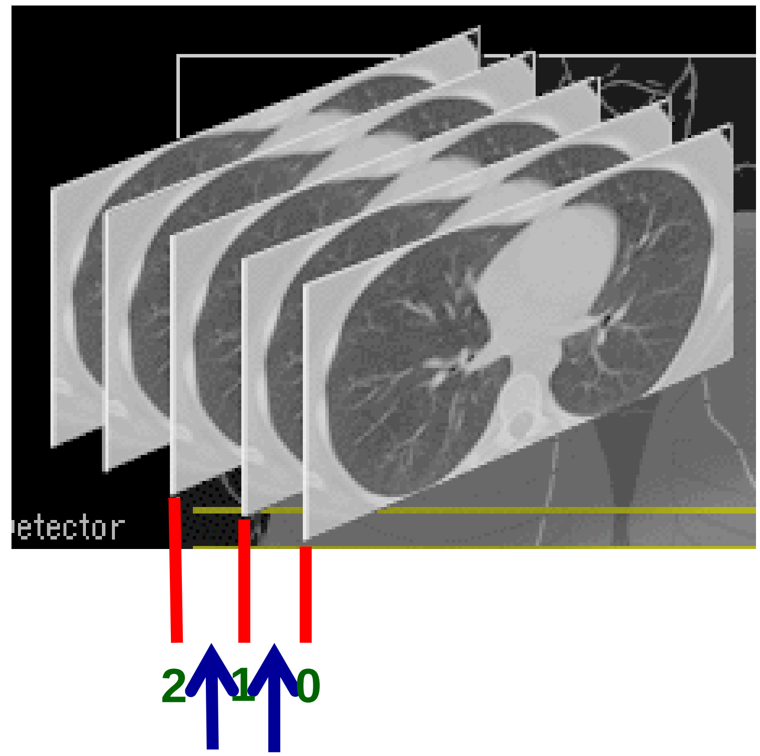
FOVY size of the body captured (in cm)

dy = FOVY / NY (pixel size, in cm)

NX number of pixels (typically 512)

FOVX size of the body captured (in cm)

dx = FOVX / NX (pixel size, in cm)



th = slice thickness (cm or mm)

$$S_{CT} = \begin{bmatrix} dx & 0 & 0 \\ 0 & dy & 0 \\ 0 & 0 & th \end{bmatrix}$$

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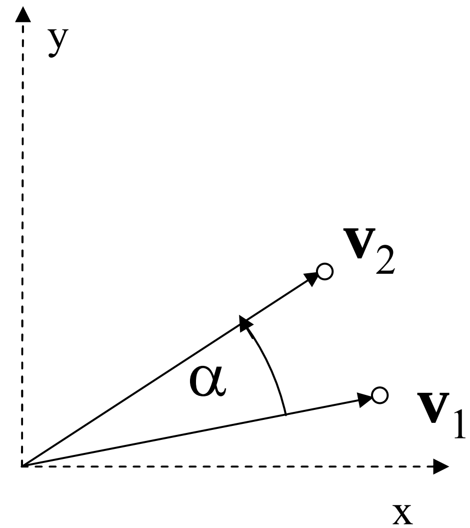
Rotation

$$\mathbf{v}_2 = \mathbf{R}_{\alpha}\mathbf{v}_1$$

$$\mathbf{v}_1 = \mathbf{R}_{-\alpha}\mathbf{v}_2$$



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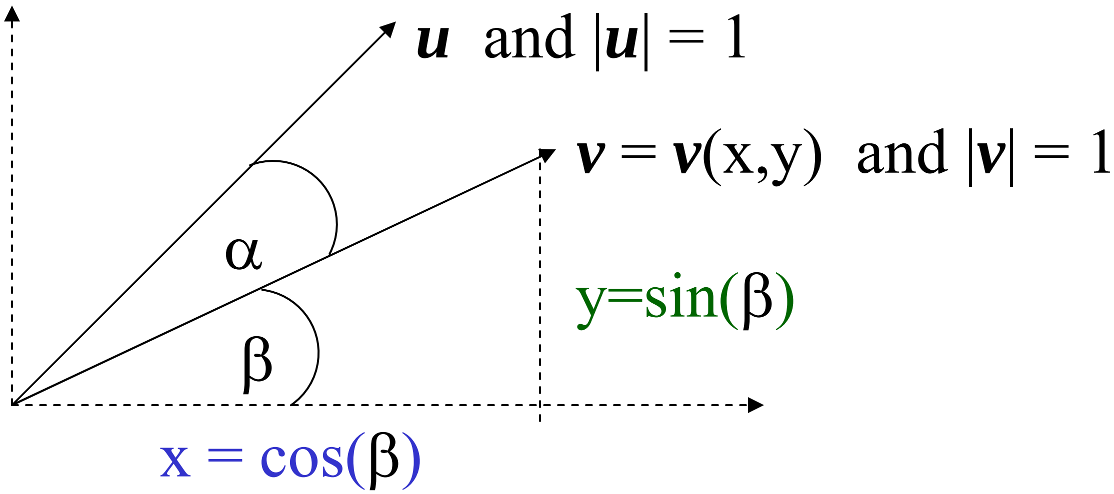
Rotation Matrix



$$\bar{\boldsymbol{u}} = \begin{bmatrix} \cos(\alpha+\beta) \\ \sin(\alpha+\beta) \end{bmatrix} = \begin{bmatrix} \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \\ \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \end{bmatrix} =$$

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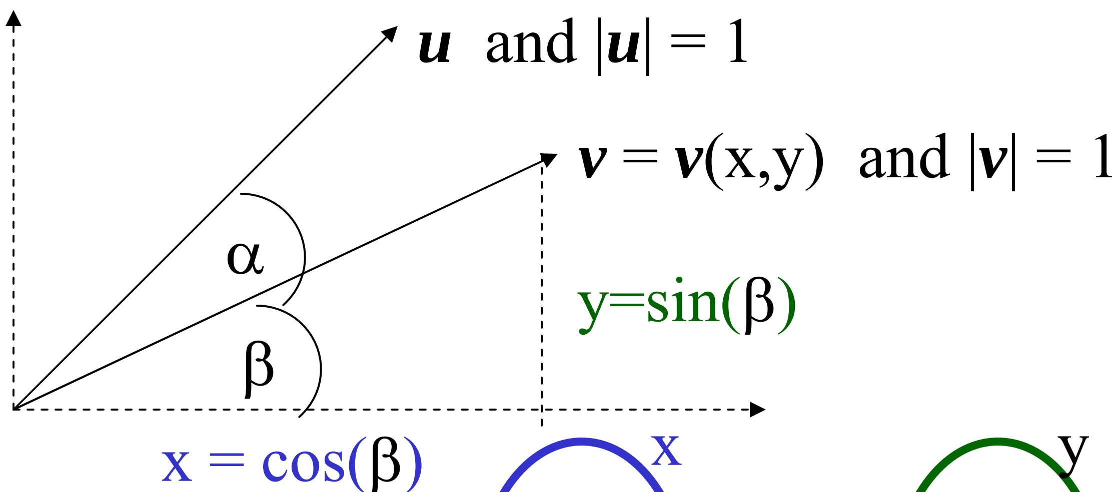
Rotation Matrix



$$\bar{\boldsymbol{u}} = \begin{bmatrix} \cos(\alpha+\beta) \\ \sin(\alpha+\beta) \end{bmatrix} = \begin{bmatrix} \cos(\alpha)\overset{x}{\cos(\beta)} - \sin(\alpha)\overset{y}{\sin(\beta)} \\ \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \end{bmatrix}$$

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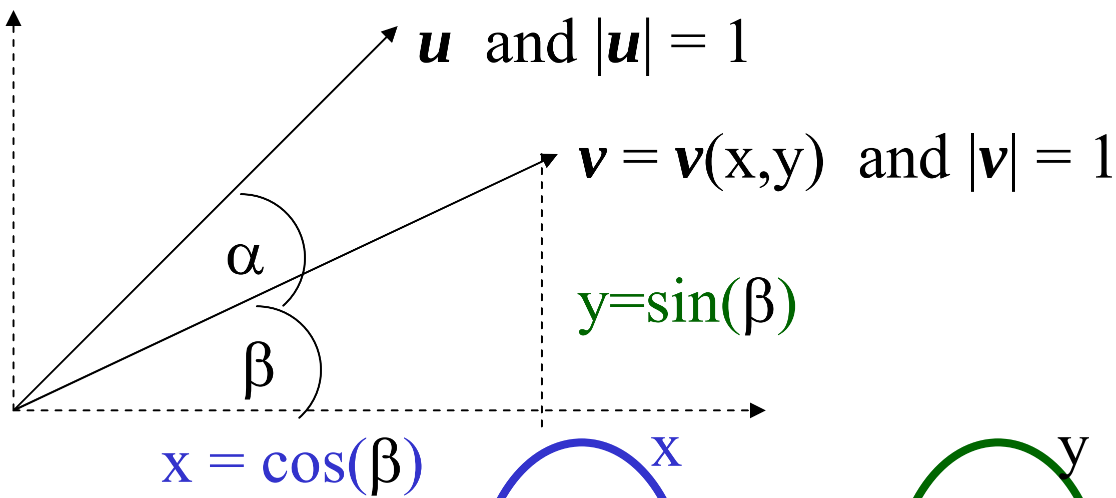
Rotation Matrix



$$\bar{\boldsymbol{u}} = \begin{bmatrix}\cos(\alpha+\beta)\\ \sin(\alpha+\beta)\end{bmatrix} = \begin{array}{c} \qquad\qquad x \qquad\qquad\qquad\qquad\qquad y \\ \begin{bmatrix}\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)\\ \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)\end{bmatrix}\end{array} =$$

$$\begin{bmatrix}\cos(\alpha)\,x - \sin(\alpha)\,y\\ \sin(\alpha)\,x + \cos(\alpha)\,y\end{bmatrix} = \begin{bmatrix}\cos(\alpha) & -\sin(\alpha)\\ \sin(\alpha) & \cos(\alpha)\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} \quad \bar{\boldsymbol{v}}$$

$$\boxed{R_\alpha = \begin{bmatrix}\cos(\alpha) & -\sin(\alpha)\\ \sin(\alpha) & \cos(\alpha)\end{bmatrix}}$$

$$\boldsymbol{u} = \mathbf{R}_\alpha\, \boldsymbol{v}$$

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Inverse of Rotation Matrix

$$R_\alpha = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \rightarrow R_{-\alpha} = \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) \\ \sin(-\alpha) & \cos(-\alpha) \end{bmatrix} =$$

$$R_{-\alpha} = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

$R_\alpha^{-1} = R_{-\alpha}$ The proof: $R_\alpha * R_{-\alpha} = I$

Series of Rotations

$$R_{(\alpha+\beta+\gamma)} = R_\alpha R_\beta R_\chi$$

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3D Rotation Matrices

Rotation by α around z axis

$$R_\alpha = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{v}_2 = R_\alpha \boldsymbol{v}_1$$

When applied on vector $\boldsymbol{v}_1$, it does not change z coordinate. Try it!

Rotation around all axes

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \qquad R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} \qquad R_z(\gamma) = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example: $\mathbf{R}_{60} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$

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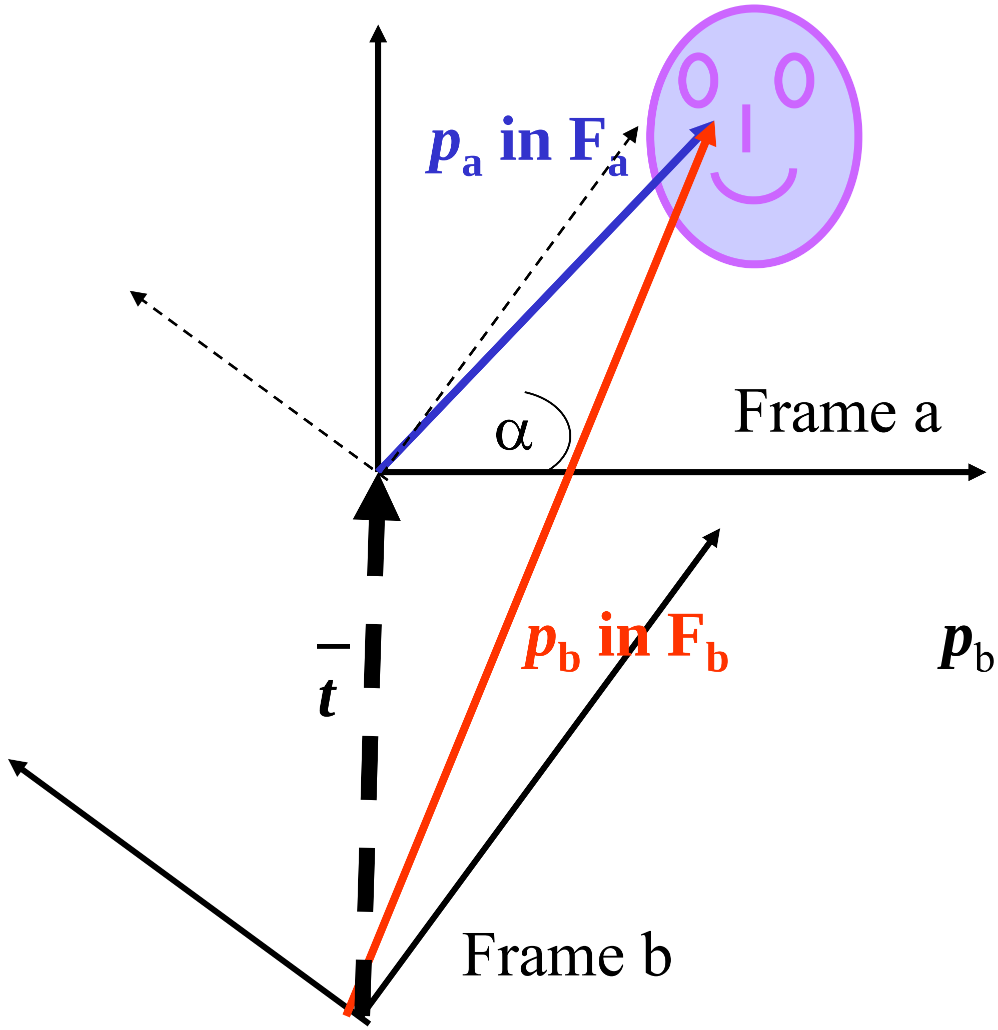
Passage from reference system to another



$$\boldsymbol{p}_{\mathrm{b}} = \mathrm{RS}\boldsymbol{p}_{\mathrm{a}} + \boldsymbol{t}$$

(1) scale, (2) rotate, and (3) translate to go from F_a to F_b

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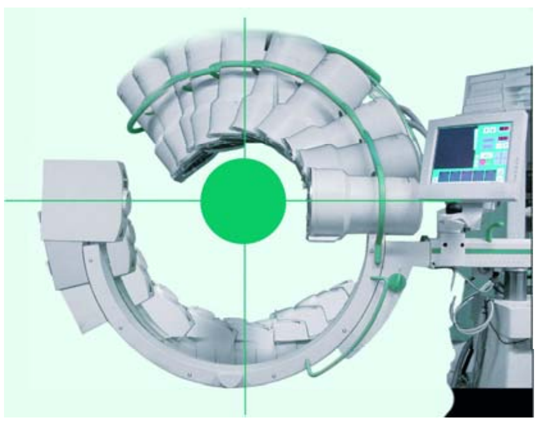
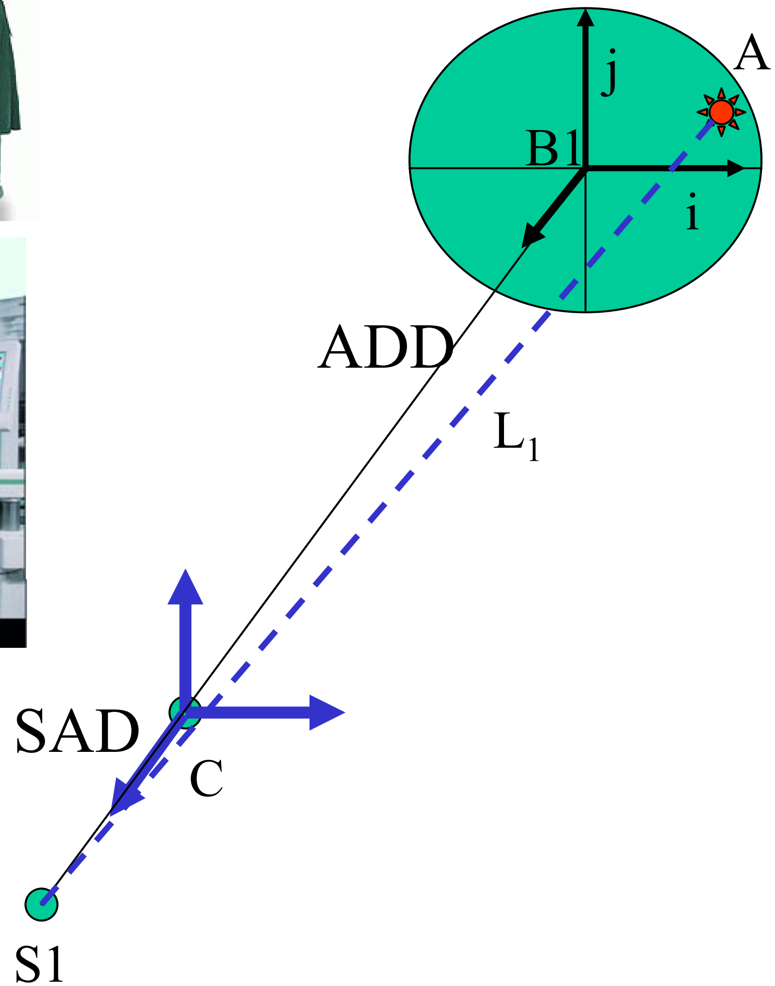
C-arm fluoroscopy





$A_{1pix} = (i_1, j_1, 0)$

Pixel size =dx,dy

$$S = \begin{vmatrix} dx & 0 & 0 \\ 0 & dy & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$B_1 = [0,0,-ADD]$

$S_1 = [0,0,SAD]$

$A_1 = B_1 + SA_{1pix}$

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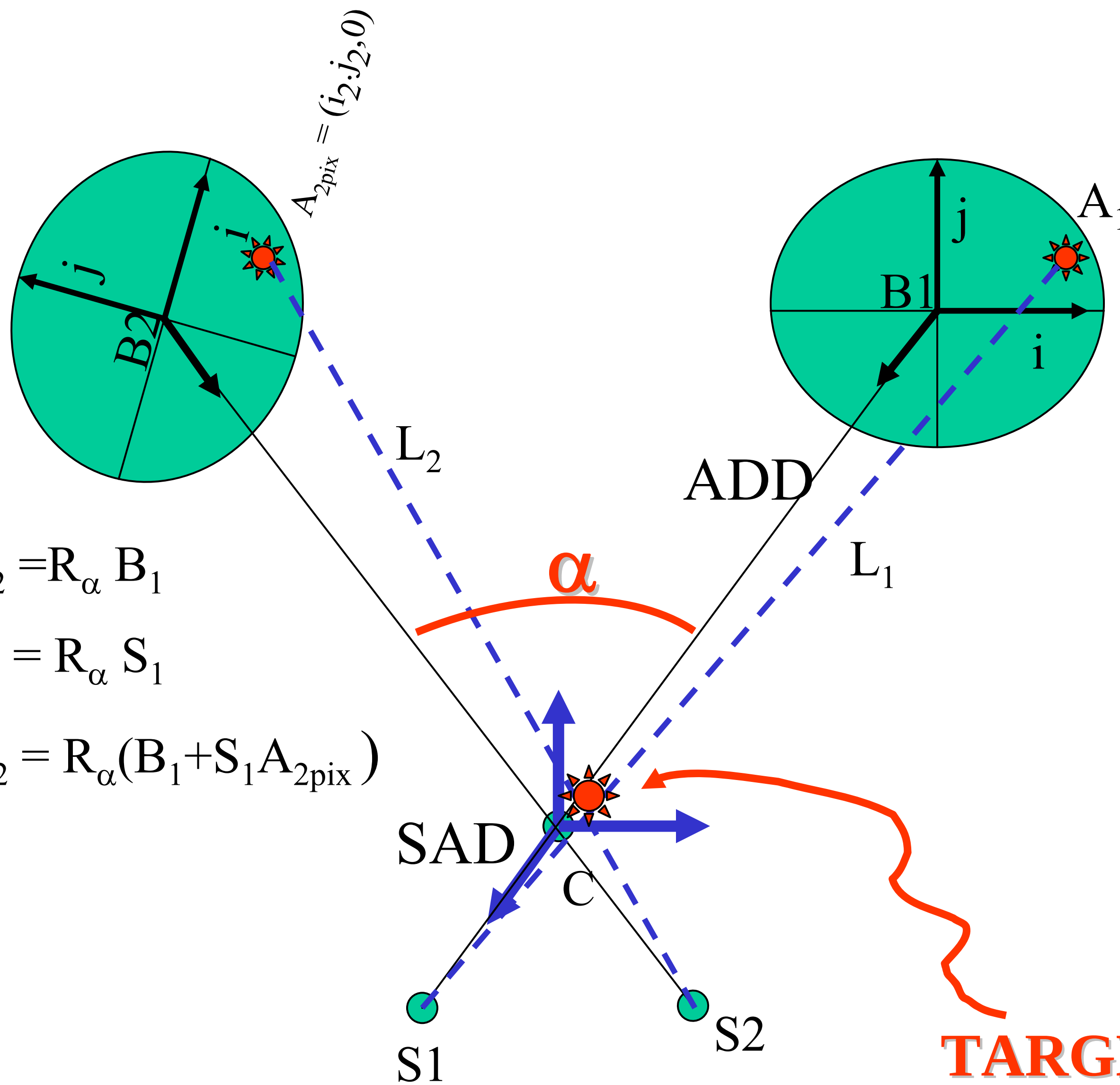
C-arm fluoroscopy



$B_2 = R_\alpha B_1$

$S_2 = R_\alpha S_1$

$A_2 = R_\alpha(B_1 + S_1 A_{2pix})$

Pixel size =dx,dy

$$S = \begin{vmatrix} dx & 0 & 0 \\ 0 & dy & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$B_1 = [0,0,-ADD]$

$S_1 = [0,0,SAD]$

$A_1 = B_1 + SA_{1pix}$

TARGET = L_1 x L_2

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Change of reference systems backwards (inverse transform)

$$\boldsymbol{p}_b = RS\boldsymbol{p}_a + \boldsymbol{t} \qquad \text{//subtract } \boldsymbol{t}$$

$$\boldsymbol{p}_b - \boldsymbol{t} = RS\boldsymbol{p}_a \qquad \text{//left multiply by } R^{-1}$$

$$R^{-1}(\boldsymbol{p}_b - \boldsymbol{t}) = S\boldsymbol{p}_a \quad \text{//left multiply by } S^{-1}$$

$$S^{-1}R^{-1}(\boldsymbol{p}_b - \boldsymbol{t}) = \boldsymbol{p}_a$$

Slight problem: mixes translation vector with matrices…
Solution: make ***t*** translation vector appear as a matrix 'T' in multiplications. Then the equations will read as:

$$\boldsymbol{p}_b = (T(R(S\boldsymbol{p}_a)))$$

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Introduce 4x4 Translation Matrix

$$T = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad v = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Padding with 1 (If I padded with 0 then inverse would not exist!)

$$Tv = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x+d_x \\ y+d_y \\ z+d_z \\ 1 \end{bmatrix}$$

Problem: rotation and scaling matrices must also be padded, so that we can multiply all 4x4 matrices.

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Homogeneous Matrices and Translation Matrix

Rotation matrix by α around z axis

$$R_\alpha = \begin{bmatrix} \cos(a) & -\sin(a) & 0 & 0 \\ \sin(a) & \cos(a) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translation matrix

$$T = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scaling matrix

$$S = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{v} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Padding all vectors with 1

$$\boldsymbol{p}_b = (T(R(S\boldsymbol{p}_a)))$$

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Series of transformations

Inverse transformation:

$Ap=q$ //apply A on p

$A^{-1}Ap = A^{-1}q$ //mult both sides from left by A^{-1}

$p=A^{-1}q$ //because $A^{-1}A=I$

Series of transformations

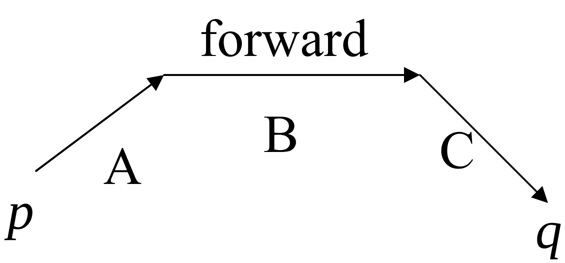
$CBAp=q$ //Apply A,B, then C on p

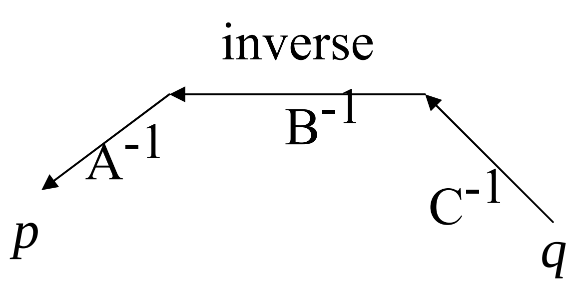
$(C\ (B\ (Ap))) = q$ //use associative property

Inverse of series of transformations

$CBAp=q$ //Apply A,B, then C on p

$A^{-1}B^{-1}C^{-1}q=p$ //multiply by inverse from left





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Inverse of coordinate transformations

$$\boxed{\boldsymbol{p}_\mathrm{b} = \mathrm{TRS}\boldsymbol{p}_\mathrm{a}}$$

Multiply from the left by T^{-1}

$$\mathrm{T}^{-1}\,\boldsymbol{p}_\mathrm{b} = \mathrm{RS}\boldsymbol{p}_\mathrm{a}$$

Multiply from the left by R^{-1}

$$\mathrm{R}^{-1}\mathrm{T}^{-1}\,\boldsymbol{p}_\mathrm{b} = \mathrm{S}\boldsymbol{p}_\mathrm{a}$$

Multiply from the left by S^{-1}

$$\boxed{\mathrm{S}^{-1}\mathrm{R}^{-1}\mathrm{T}^{-1}\boldsymbol{p}_\mathrm{b} = \boldsymbol{p}_\mathrm{a}}$$

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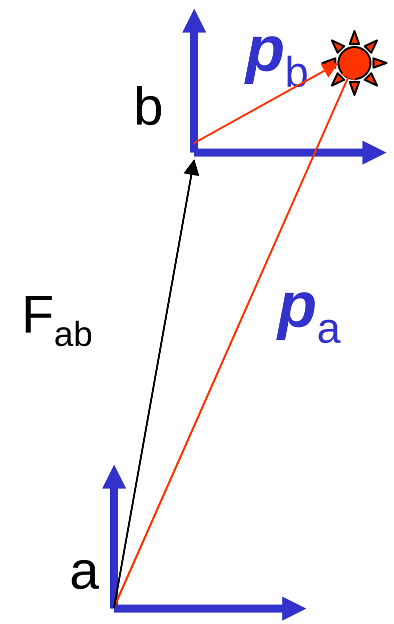
Reference frame transformations

$$\boldsymbol{p}_b = T*R*S* \boldsymbol{p}_a$$

$$\boldsymbol{p}_b = F_{ab}\boldsymbol{p}_a$$



F_{ab} is often called frame transformation or shortly frame

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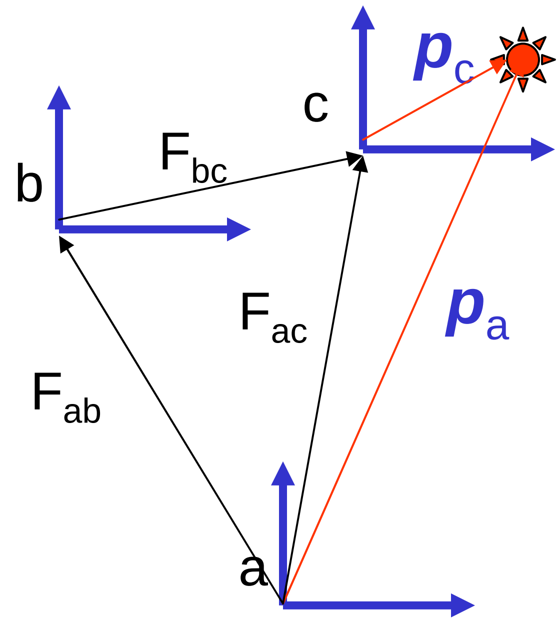
Series of reference frame transformation

Most real systems involve complicated series of frames, where we traverse from frame to frame…

$$\boldsymbol{p}_c = \underbrace{F_{bc}F_{ab}}_{}\boldsymbol{p}_a \quad F_{ac}$$



All frames must be known: some of them are measured/calculated during the procedure, some of them are pre-operatively known through “calibration”.

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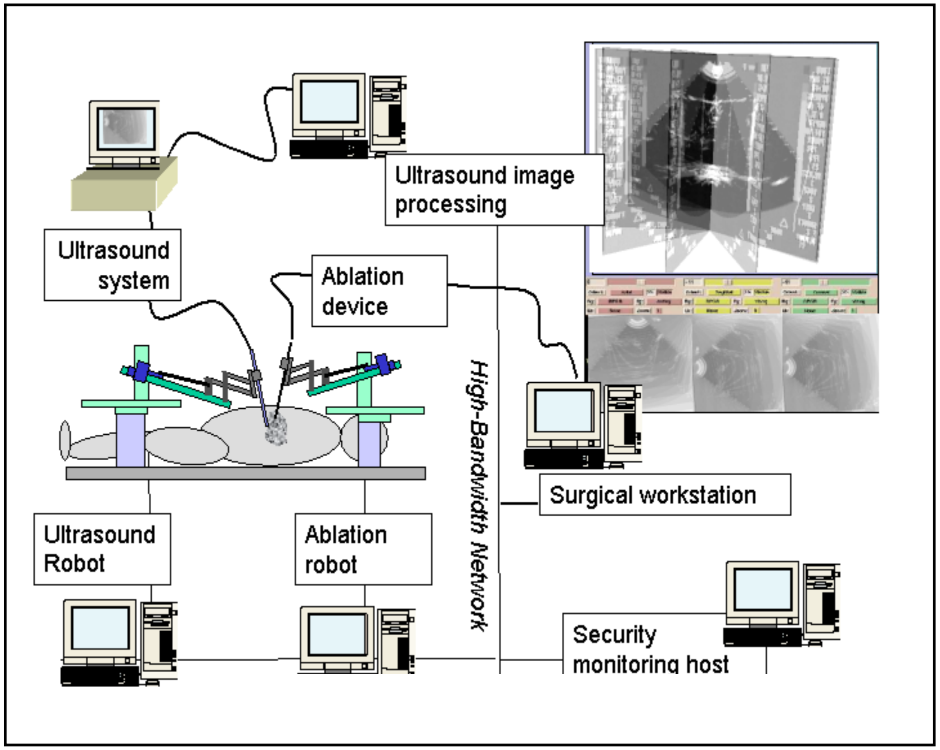
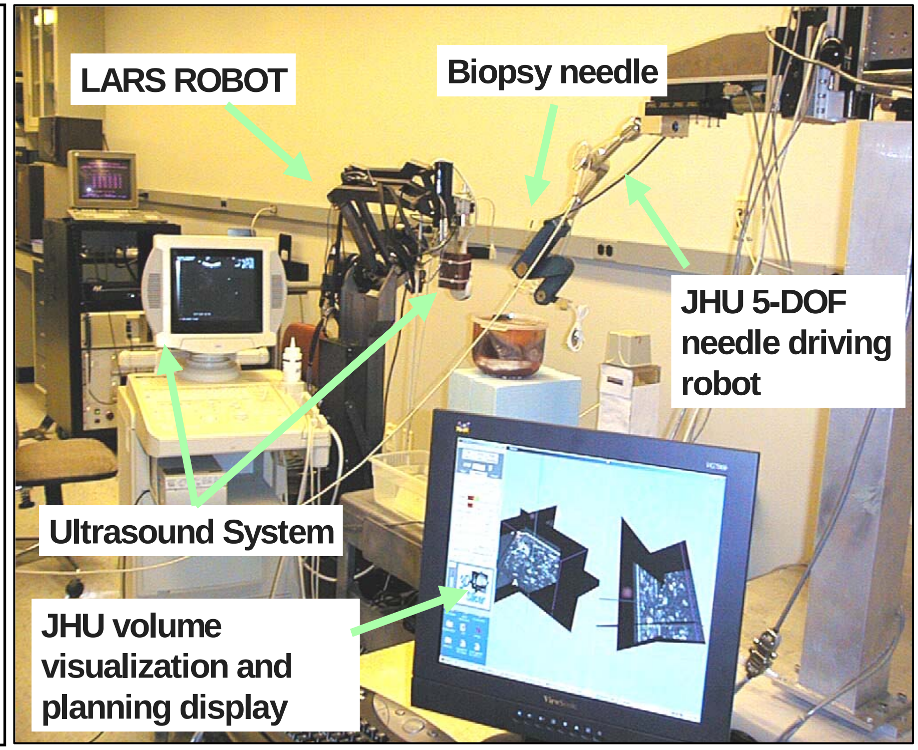
Example: Robot-Assisted Ultrasound-Guided Liver Surgery





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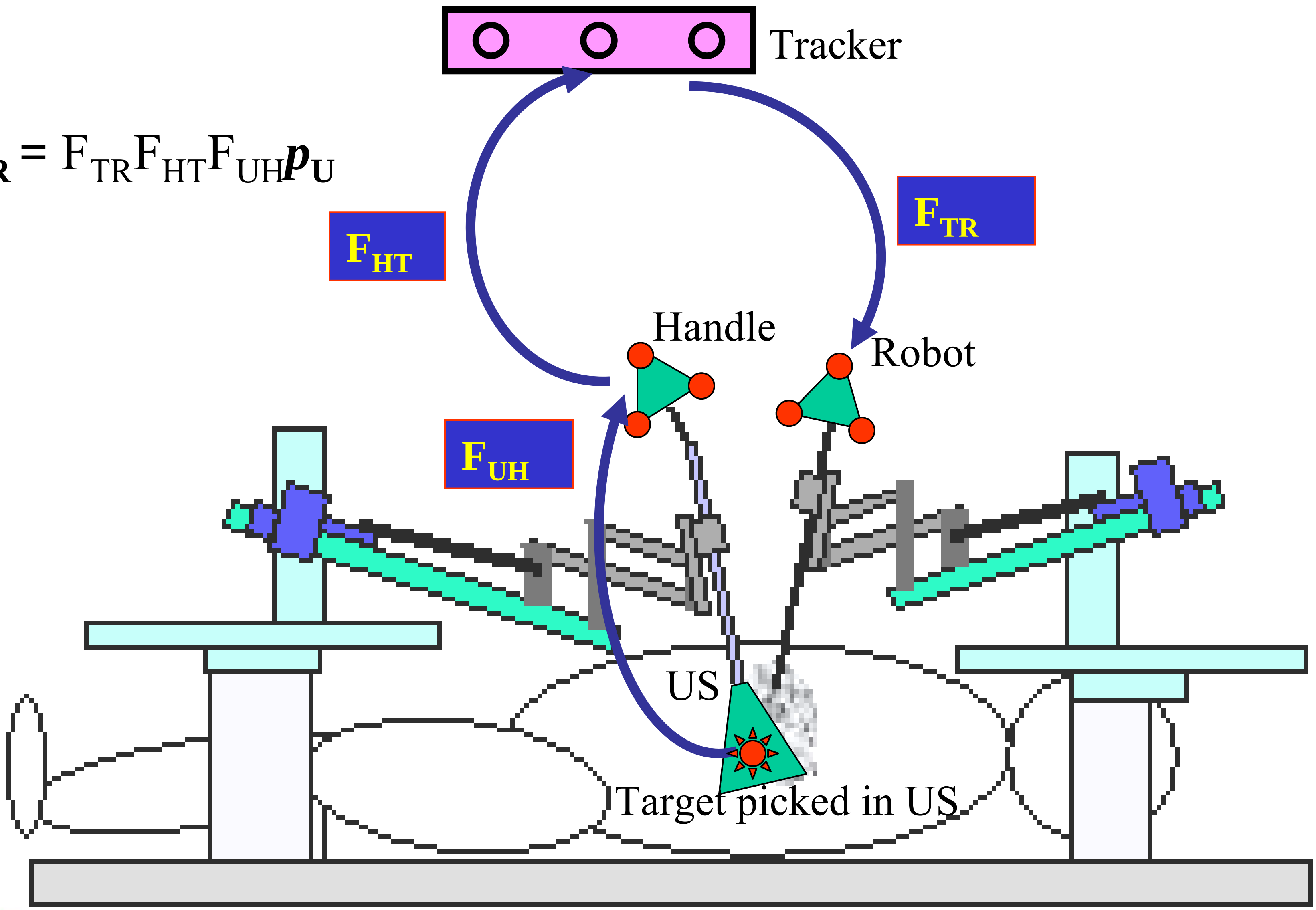
Example cont'd: some frames in the tracked dual-arm robot system

$$p_R = F_{TR} F_{HT} F_{UH} p_U$$



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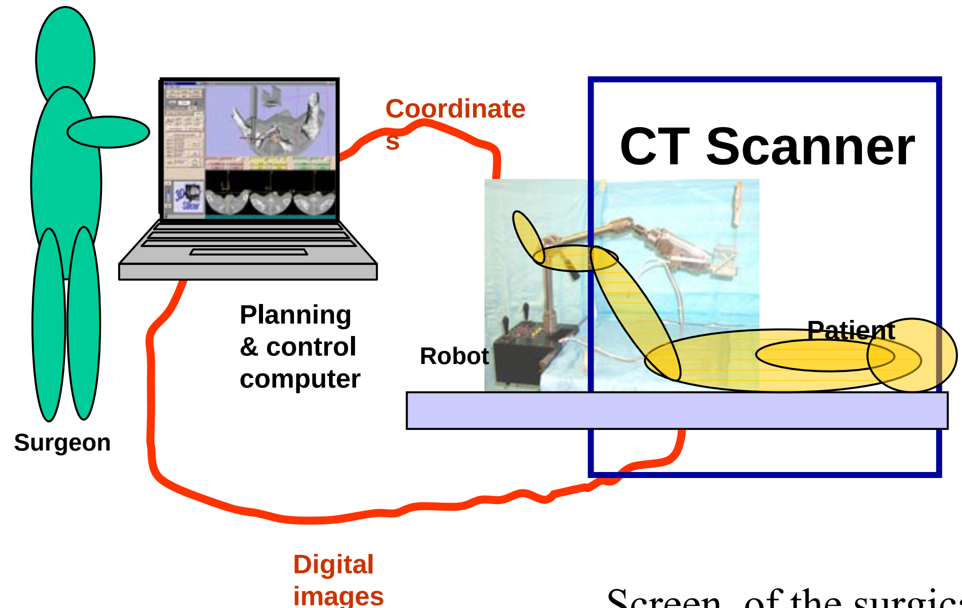
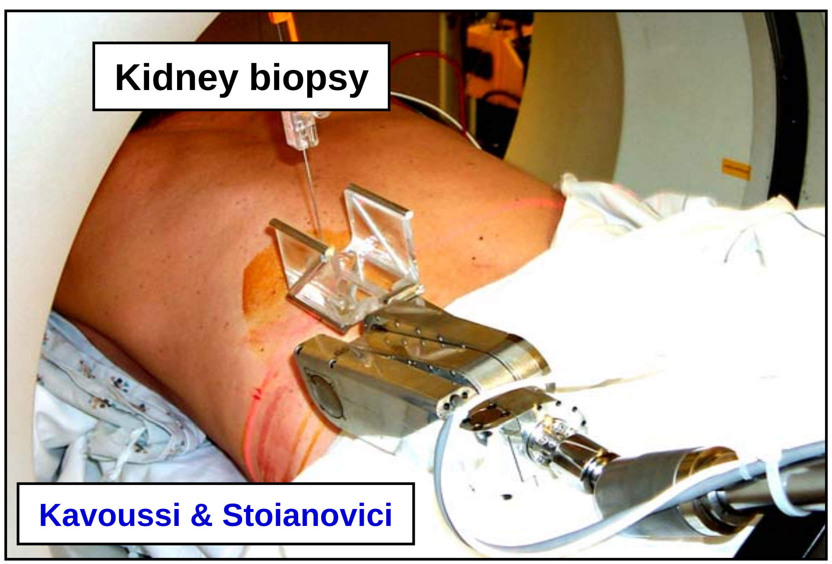
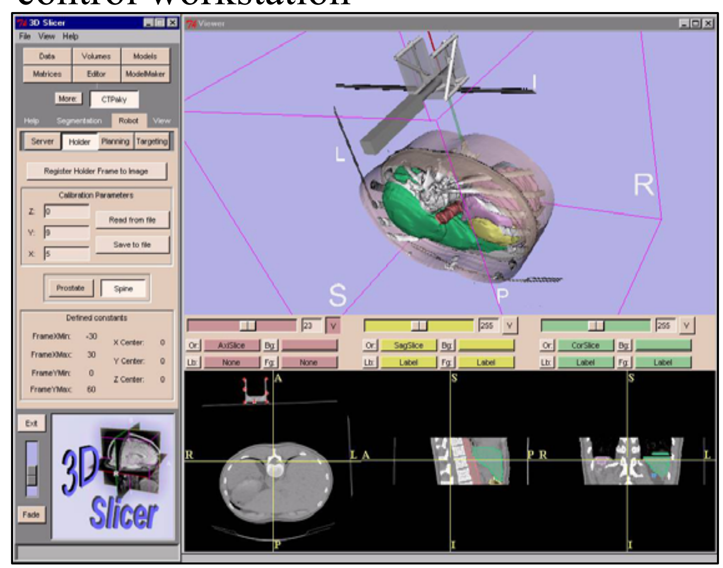
Example: CT-guided Needle Placement with Robot



Patient in the scanner with robot



Screen of the surgical planning and control workstation



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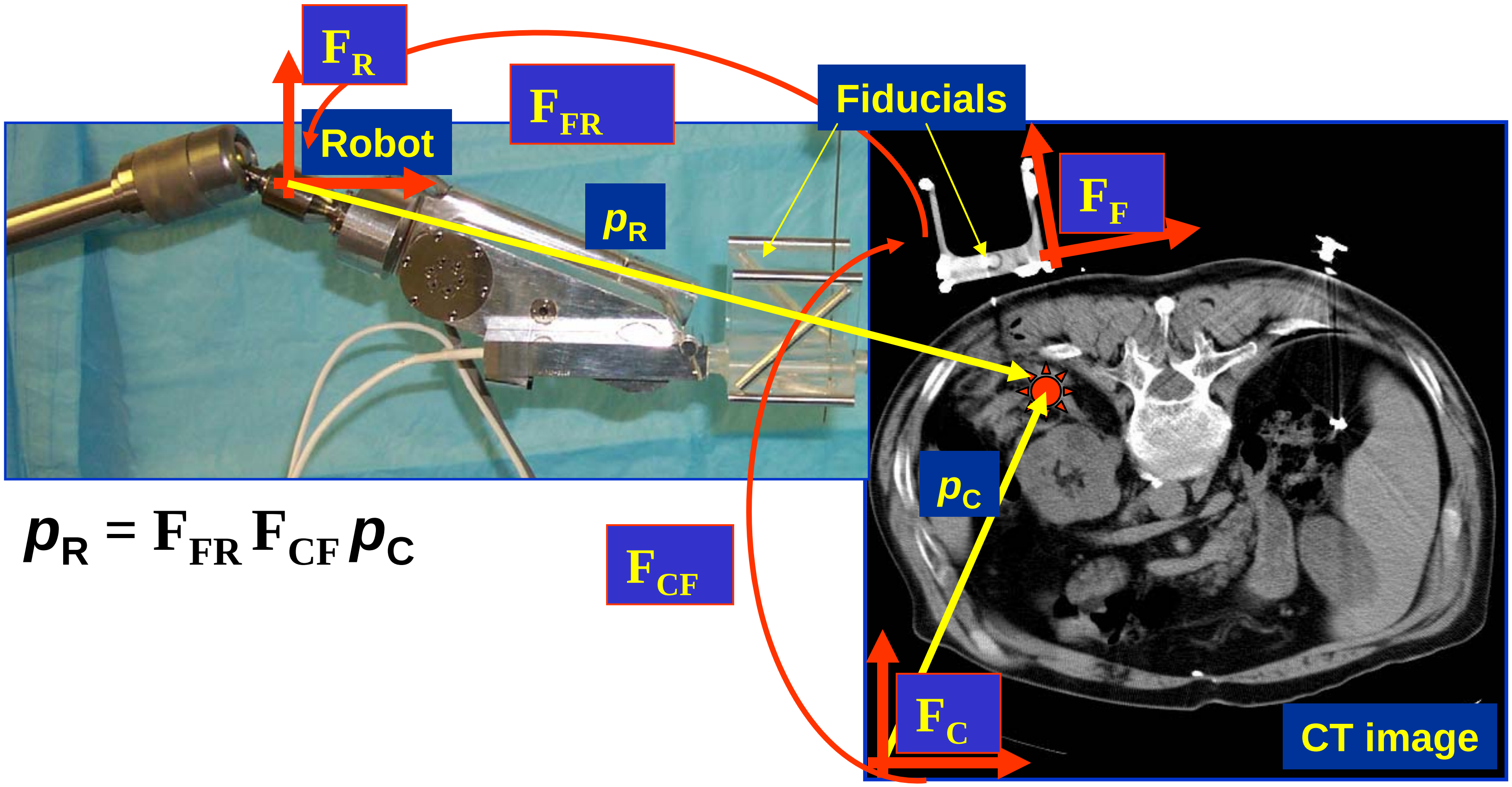
Example cont'd: Frames



$$\boldsymbol{p}_R = F_{FR}\, F_{CF}\, \boldsymbol{p}_C$$

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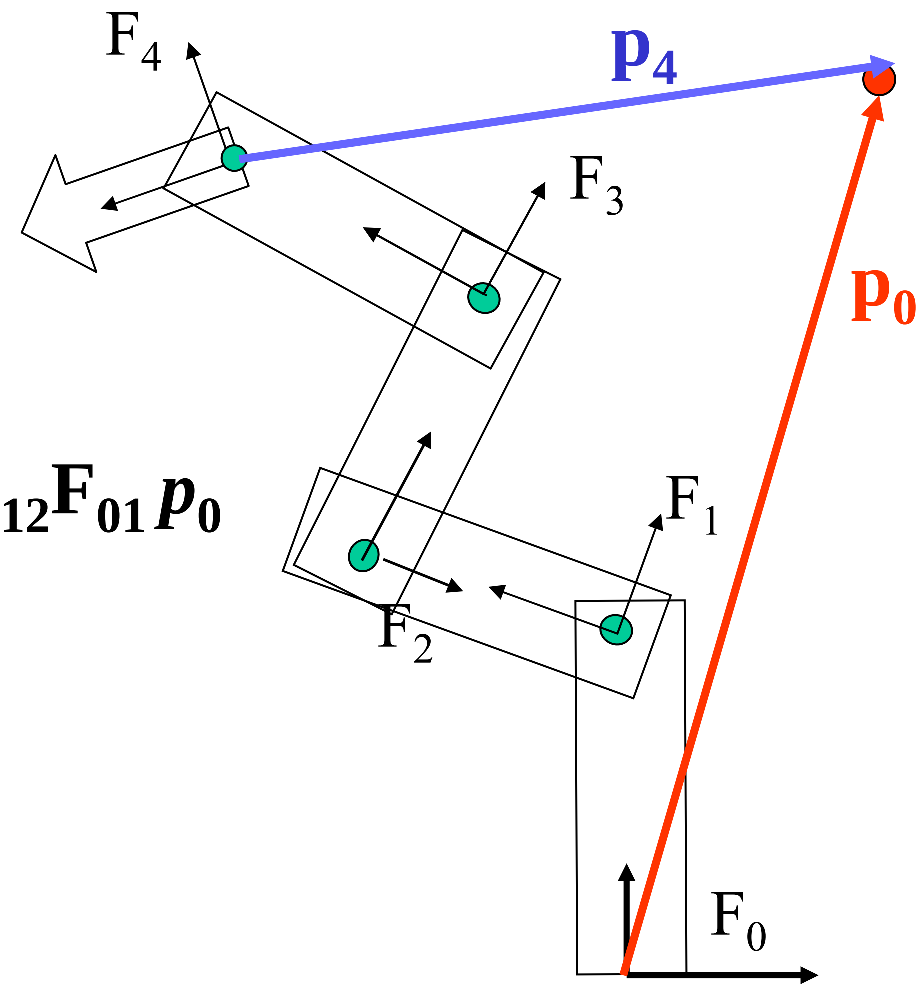
Example: Multi-Joint Serial Robots

4-joint serial robot



$$\boldsymbol{p}_4 = \mathbf{F}_{34}\mathbf{F}_{23}\,\mathbf{F}_{12}\mathbf{F}_{01}\,\boldsymbol{p}_0$$

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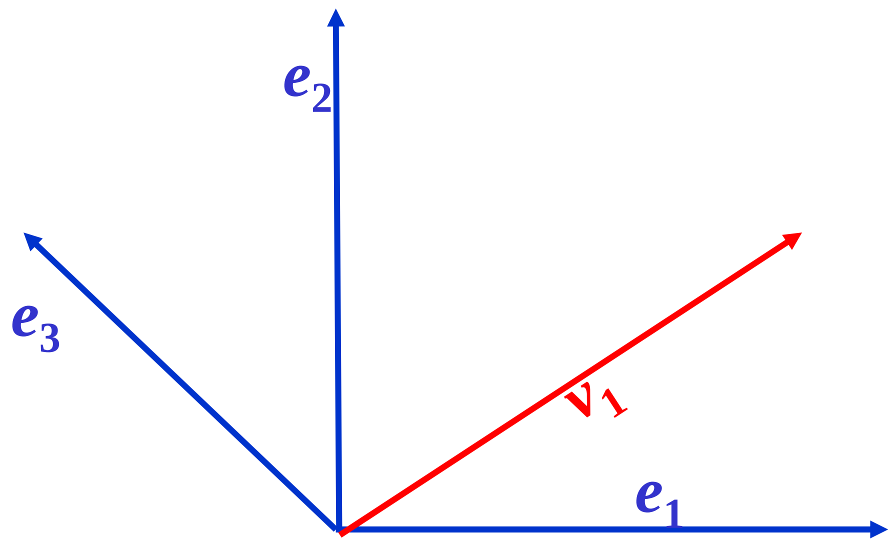
e1 e2 e3 orthonormal base



Let $e_1\, e_2\, e_3$ be orthonormal base vectors…
Let v_1 be an arbitrary vector…

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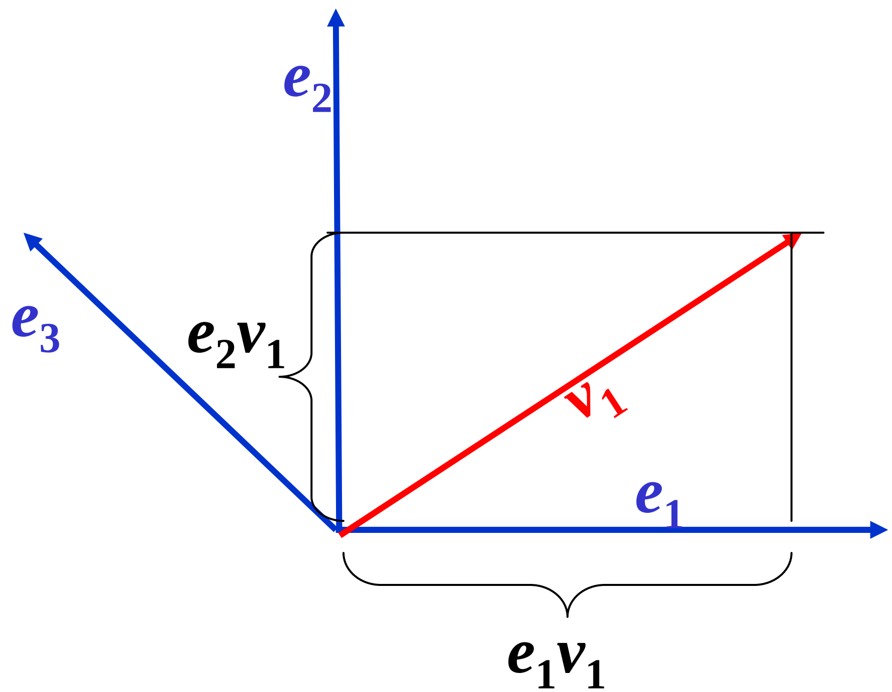
Express v_1 with e_1 e_2 e_3 base vectors



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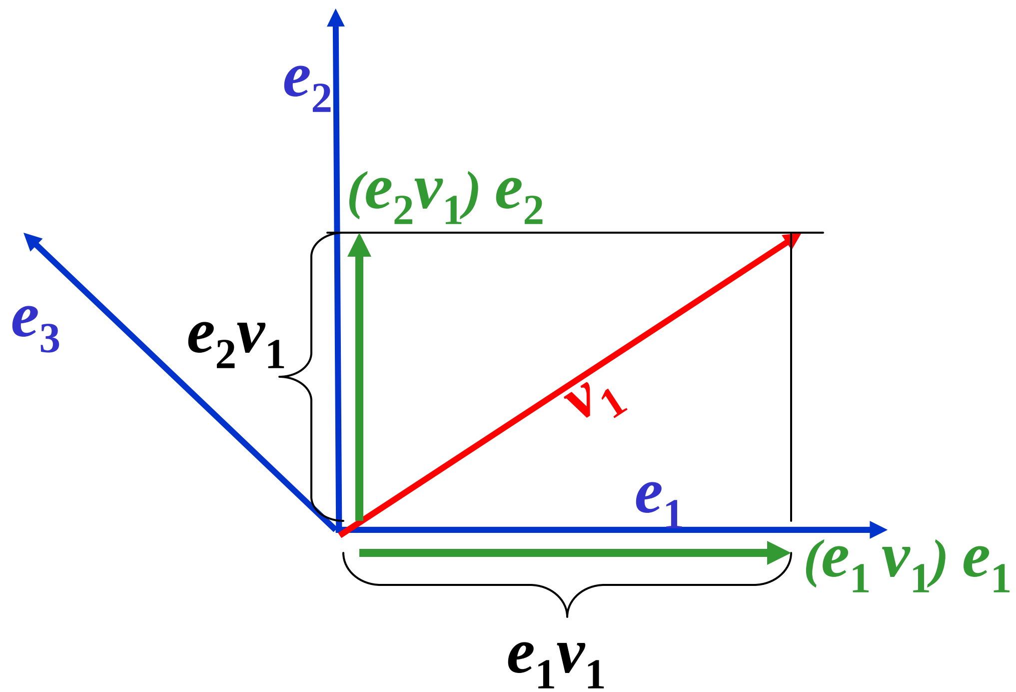
Express v_1 with $e_1\, e_2\, e_3$ base vectors



$$v_1 = (e_1\, v_1)\, e_1 + (e_2\, v_1)\, e_2 + (e_3\, v_1)\, e_3$$

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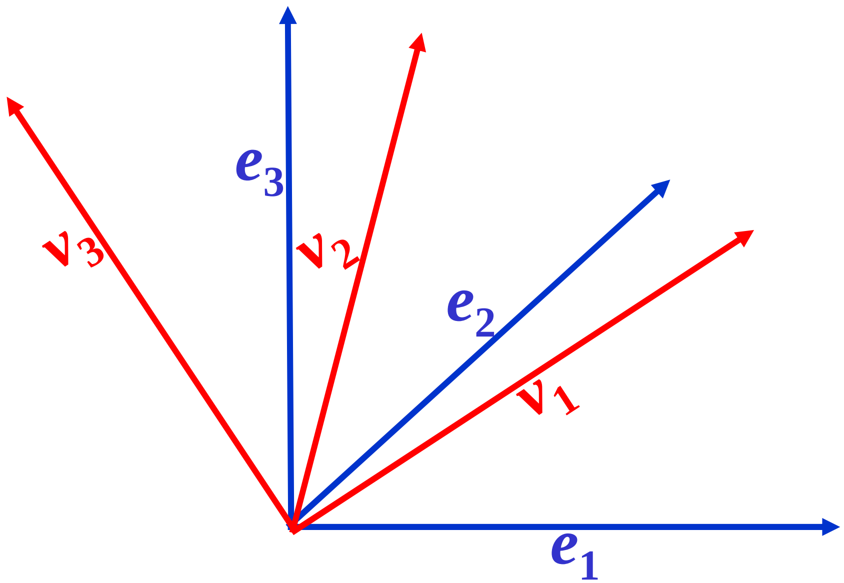
Express $v_1\ v_2\ v_3$ base with $e_1\ e_2\ e_3$ base vectors



$$v_1 = (e_1\, v_1)\, e_1 + (e_2\, v_1)\, e_2 + (e_3\, v_1)\, e_3$$
$$v_2 = (e_1\, v_2)\, e_1 + (e_2\, v_2)\, e_2 + (e_3\, v_2)\, e_3$$
$$v_3 = (e_1\, v_3)\, e_1 + (e_2\, v_3)\, e_2 + (e_3\, v_3)\, e_3$$

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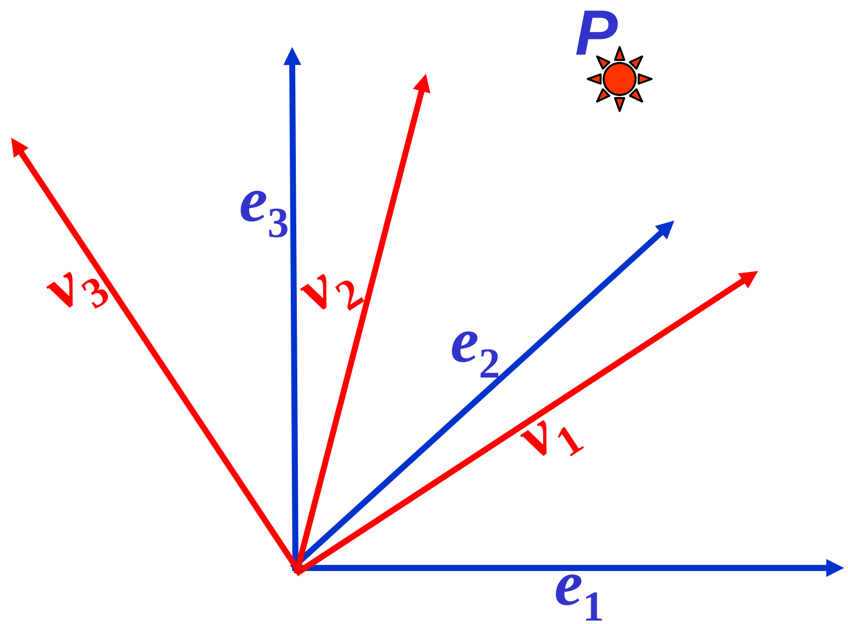
Express P in both coordinate systems



(Expressed $v_1\, v_2\, v_3$ with $(e_1\, e_2\, e_3)$ orthonormal base vectors)

$$\boldsymbol{v}_1 = (\boldsymbol{e}_1\, \boldsymbol{v}_1)\, \boldsymbol{e}_1 + (\boldsymbol{e}_2\, \boldsymbol{v}_1)\, \boldsymbol{e}_2 + (\boldsymbol{e}_3\, \boldsymbol{v}_1)\, \boldsymbol{e}_3$$

$$\boldsymbol{v}_2 = (\boldsymbol{e}_1\, \boldsymbol{v}_2)\, \boldsymbol{e}_1 + (\boldsymbol{e}_2\, \boldsymbol{v}_2)\, \boldsymbol{e}_2 + (\boldsymbol{e}_3\, \boldsymbol{v}_2)\, \boldsymbol{e}_3$$

$$\boldsymbol{v}_3 = (\boldsymbol{e}_1\, \boldsymbol{v}_3)\, \boldsymbol{e}_1 + (\boldsymbol{e}_2\, \boldsymbol{v}_3)\, \boldsymbol{e}_2 + (\boldsymbol{e}_3\, \boldsymbol{v}_3)\, \boldsymbol{e}_3$$

Express P in both coordinate systems…

$\boldsymbol{P}$ = r $\boldsymbol{e}_1$+ s $\boldsymbol{e}_2$ + t$\boldsymbol{e}_3$ (r, s, t are coordinate values)

$\boldsymbol{P}$ = x $\boldsymbol{v}_1$+ y$\boldsymbol{v}_2$ + z$\boldsymbol{v}_3$ (x, y, z are coordinate values)

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Generalized rotation matrix

Replace v_1, v_2 and v_3 with the "long forms" we just derived

$$\begin{cases} v_1 = (e_1 v_1)\, e_1 + (e_2 v_1)\, e_2 + (e_3 v_1)\, e_3 \\ v_2 = (e_1 v_2)\, e_1 + (e_2 v_2)\, e_2 + (e_3 v_2)\, e_3 \\ v_3 = (e_1 v_3)\, e_1 + (e_2 v_3)\, e_2 + (e_3 v_3)\, e_3 \end{cases}$$

$\boldsymbol{P} = \mathrm{r}\, \boldsymbol{e_1} + \mathrm{s}\, \boldsymbol{e_2} + \mathrm{t}\boldsymbol{e_3}$ (r, s, t are coordinate values)

$\boldsymbol{P} = \mathrm{x}\, \boldsymbol{v_1} + \mathrm{y}\boldsymbol{v_2} + \mathrm{z}\boldsymbol{v_3}$ (x, y, z are coordinate values)

$$\begin{aligned} P = {} & x(e_1 v_1)\, e_1 + x(e_2 v_1)\, e_2 + x(e_3 v_1)\, e_3 + \\ & y(e_1 v_2)\, e_1 + y(e_2 v_2)\, e_2 + y(e_3 v_2)\, e_3 + \\ & z(e_1 v_3)\, e_1 + z(e_2 v_3)\, e_2 + z(e_3 v_3)\, e_3 \end{aligned}$$

Group the e_1, e_2 and e_3 members …

$$\begin{aligned} P = {} & x(e_1 v_1)\, e_1 + y(e_1 v_2)\, e_1 + z(e_1 v_3)\, e_1 + \\ & x(e_2 v_1)\, e_2 + y(e_2 v_2)\, e_2 + z(e_2 v_3)\, e_2 + \\ & x(e_3 v_1)\, e_3 + y(e_3 v_2)\, e_3 + z(e_3 v_3)\, e_3 \end{aligned}$$

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Generalized rotation matrix

Group the scale factors …

$$P = (x(e_1 v_1) + y(e_1 v_2) + z(e_1 v_3))\ e_1 + (x(e_2 v_1) + y(e_2 v_2) + z(e_2 v_3))\ e_2 + (x(e_3 v_1) + y(e_3 v_2) + z(e_3 v_3))\ e_3$$

Rearrange to column vector format….

$$\begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} x(e_1 v_1) + y(e_1 v_2) + z(e_1 v_3) \\ x(e_2 v_1) + y(e_2 v_2) + z(e_2 v_3) \\ x(e_3 v_1) + y(e_3 v_2) + z(e_3 v_3) \end{bmatrix}$$

*Recognize matrix*vector product….*

$$P \text{ in } (rst) \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} (e_1 v_1) & (e_1 v_2) & (e_1 v_3) \\ (e_2 v_1) & (e_2 v_2) & (e_2 v_3) \\ (e_3 v_1) & (e_3 v_2) & (e_3 v_3) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} P \text{ in } (xyz)$$

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Generalized rotation matrix

Direction cosines

$$R = \begin{bmatrix} (e_1 v_1) & (e_1 v_2) & (e_1 v_3) \\ (e_2 v_1) & (e_2 v_2) & (e_2 v_3) \\ (e_3 v_1) & (e_3 v_2) & (e_3 v_3) \end{bmatrix}$$

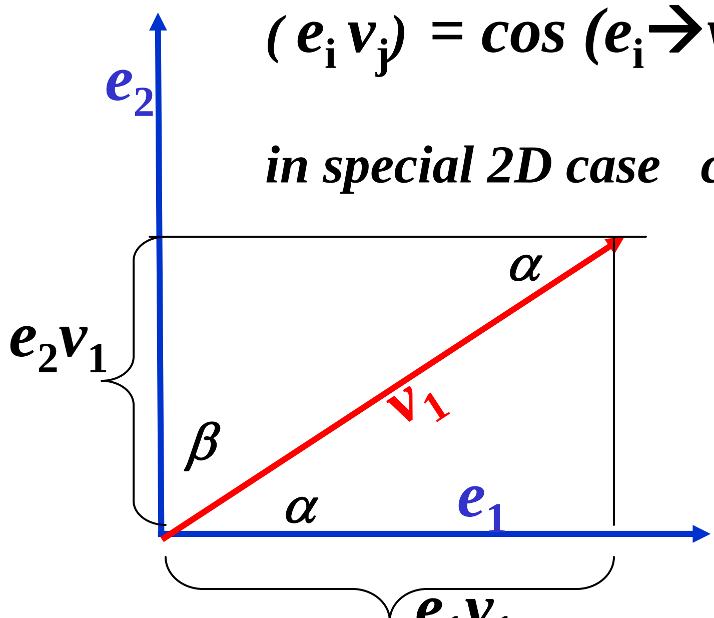
where

$$(e_1 v_1) = \cos(e_1 \rightarrow v_1) = \cos(\alpha)$$

$$(e_2 v_1) = \cos(e_2 \rightarrow v_1) = \cos(\beta)$$

$$(e_i v_j) = \cos(e_i \rightarrow v_j) = \cos(\beta_{ij})$$

in special 2D case $\cos(b) = \sin(a)$



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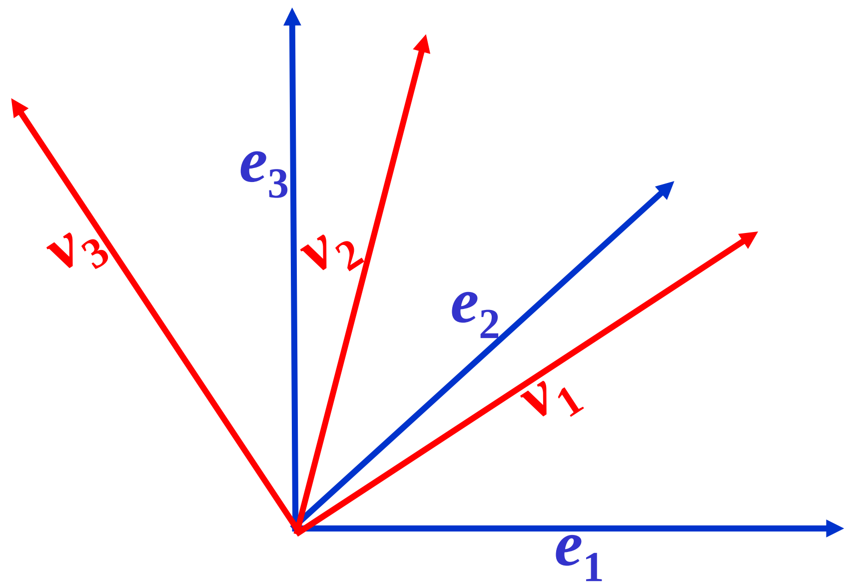
Generalized rotation matrix

$$\boldsymbol{R} = \begin{bmatrix} (e_1 v_1) & (e_1 v_2) & (e_1 v_3) \\ (e_2 v_1) & (e_2 v_2) & (e_2 v_3) \\ (e_3 v_1) & (e_3 v_2) & (e_3 v_3) \end{bmatrix}$$



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