Selection, with or without ordering.

Solving counting problems is a matter of obtaining a bijection between the problem you need to solve and something you already know how to count. Often you have to combine several different techniques to solve a counting problem. There are examples where there are more than one correct way to solve a counting problem. The following problems can be solved by using the paradigm of selection (balls in a bag) with or without ordering.
How many different 5 card hands (a selection of 5 cards without ordering from a 52 card deck) are there with 4 aces?

This problem is modeled by selection without ordering.

But first consider:

How many different 4 card hands are there with 4 aces?

There is an easy solution to this problem by obtaining a bijection to the problem of selecting one card from a deck of 48 cards.

There is only one way to choose 4 aces from a deck of cards.

Once the 4 aces are removed there remains 48 cards to choose from. So the number of 5 card hands with 4 aces is 48.
How many different 5 card ordered selections from a 52 card deck are there with 4 aces?

This is modeled by selection with ordering. We have seen counting problems where selecting without ordering is solved by first solving a selection with ordering and then divide by the number of orderings. Here there is a simple solution using selection without ordering and multiplying by the number of orderings.

So as before there are 48 unordered selections. Each one can be ordered 5! ways. So the answer is $48 \times 5!$.
How many 5 card hands are there that contain 4 cards of the same value? (A four of a kind)

This is similar to the previous problem.

There are 13 possible values that could be the 4 of a kind. So there are 13 ways to choose 4 cards that are 4 of a kind, and 48 ways to choose the 5th card. Now take the product to get 13 × 48.
A Full House is a 5 card hand with three cards of one value and two cards of another value. Here are some examples:

\[2\clubsuit, 2\diamondsuit, A\heartsuit, A\spadesuit \text{ or } 7\heartsuit, 7\diamondsuit, 9\clubsuit, 9\diamondsuit, 9\heartsuit\]

How many ways are there to select a full house from a 52 card deck?

We obtain a bijection between a full house and a string of the form

\[(P,T)\{SP1,SP2\} \{ST1,ST2,ST3}\]

Where:

1. P is the value of the pair, and T is the value of the triple. \(13 \times 12\) choices
2. SP1 and SP2 are the suits for the pair. \(\binom{4}{2}\) choices.
3. ST1,ST2,ST3 are the suits of the triple. \(\binom{4}{3}\) choices.

Note the symbols grouped in \{\} are dependent choices, but outside the \{\} they are independent.
Let’s check that we have counted correctly. A common problem is not having a proper bijection between the model and the artifact we want to count. We need to show that the string representation is in a bijection with a full house.

We can translate 2♠, 2♦, A ♦, A ♥, A ♣ into the representation (2,A){♠ ♦} {♦ ♥ ♣}, and we can verify that this represents a unique full house.
How many ways are there to select a 5 card hand that contains two distinct pairs of cards that are of the same value plus one more card of the third value.

For example:

2♣,2♦, A♥, 3♠ or 7♣,7 ♥, 9♣, 9♦, K ♠

We use this string to represent a counting scheme. It is essential that you verify that the representation maps faithfully to the counting problem.

We have:

\[ ((PA, PB), O), \{SPA1, SPA2\}, \{SPB1, SPB2\}, \{SO\} \]

Where:
1. PA is the value of the A pair, PB is the value of the B pair. There are \( \binom{13}{2} \) choices.
2. And O is the value of the other card with 11 choices.
3. SPA1,SPA2 are the suits for the pair A. \( \binom{4}{2} \) choices
4. SPB1,SPB2 are the suits for the pair B. \( \binom{4}{2} \) choices
5. SO is the suit of the other card with 4 choices.

Thus we get:

\[
\binom{13}{2} \times 11 \times \binom{4}{2} \times \binom{4}{2} \times 4
\]

ways to get 2 pairs.
How many ways are there to choose a 5 card hand such that there are 4 cards with 4 different suits, (🂫, ♦, ♥, ♣) and the fifth card could be anything.

Let's start with a simpler problem, how many ways are there to choose a 4 card hand with 4 different suits.

This problem can be modeled by using 4 different bags, or by allowing replacement, of the object that has just been selected, and ordering the selections to designate clubs, diamonds hearts and spades.

It should be clear that there are $13^4$ ways

This is known as *selection with replacement and with ordering*.

The counting formula is a direct application of the product rule, that is, the number of ways to select times from $n$ objects with replacement and with ordering is:

$$n^k$$
Back to the original problem. The impulse is to do as before and simply multiply by 48.

For example:

Suppose the 4 cards of different suit are

7 ♡, 3♢, 9♡, 2♣

and the 5th card is A♡, so that the five cards are:

7 ♡, 3♢, 9♡, 2♣, A♡

But this is identical to choosing 4 cards of different suit

7 ♡, 3♢, A♡, 2♣

and the fifth card 9♡.

The value $13^4 \times 48$ counts every hand exactly twice, so the correct answer is

$\frac{(13^4 \times 48)}{2}$
We view counting problems as selecting balls from a bag with or without ordering and with or without replacement.

<table>
<thead>
<tr>
<th>Select k balls from a bag of n labelled balls</th>
<th>With ordering</th>
<th>Without ordering</th>
</tr>
</thead>
<tbody>
<tr>
<td>With replacement</td>
<td>$n^k$</td>
<td></td>
</tr>
<tr>
<td>Without replacement</td>
<td>$\frac{n!}{(n - k)!}$</td>
<td>$\binom{n}{k} = \frac{n!}{(n - k)!k!}$</td>
</tr>
</tbody>
</table>

There is a missing entry, that is, how do we count selection with replacement and without ordering.
You get to pick a box of 10 timbits® and choose as many as you like from the choice of

Chocolate, Sugar, Plain, Glazed

The way to model this is to consider a bag with balls labelled C,S,P,G and we count the number of ways to select 10 without ordering and with replacement. We usually count this by selecting with ordering and then divide by the number of orderings. However...

Suppose the 10 choices in order are


There are $10!/ (3 \times 3!)$ ways to order these.

On the other hand suppose the choices in order are:


There are $10!/10! = 1$ way to order this choice.

It appears that our existing methods do not solve this counting problem very easily.
Consider the following seemingly unrelated problem, that of counting the number of binary strings of length 13, consisting of 10 0’s and 3 1’s.

For example: 0100010001000

We can count the total number of this type of string as

$$\frac{13!}{3!10!}$$

Now consider a bijection from binary strings to donut selections.

I claim that there is a bijective mapping from binary strings of length 13 to strings using letters C,S,P,G of length 10.


The mapping works as follows:

The 10 0’s represent timbits®, the 1’s act as dividers partitioning the zeros into 4 groups corresponding to the 4 varieties of timbits®.

What does this 0000000000111 binary string represent?
We can now complete the table as follows:

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<tbody>
<tr>
<td>With replacement</td>
<td>( n^k )</td>
<td>( \binom{n-1+k}{k} )</td>
</tr>
<tr>
<td>Without replacement</td>
<td>( \frac{n!}{(n-k)!} )</td>
<td>( \binom{n}{k} = \frac{n!}{(n-k)!k!} )</td>
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The Pigeon Hole Principle

If there are \( n \) pigeons, that all must sleep in a pigeon hole, and \( n-1 \) pigeon holes, then there is at least one pigeon hole where 2 pigeons sleep.

This should be obvious! Mathematicians give it a name because it is a useful counting tool.

Can we find two people living in the G.T.A. that have exactly the same number of strands of hair on their heads?

The answer is YES! And we can prove it using the pigeon hole principle.
The population of the G.T.A is more than 6 million. Science tells us that nobody has more that 500,000 strands of hair on their heads.

To solve the problem using the pigeon hole principle we imagine 500,000 pigeon holes labelled from 1, ..., 500,000 and then imagine each resident of the G.T.A. entering the pigeon hole labelled with the number of strands of hair on their head. Since 6 million is greater than 500,000 we deduce that there will be at least one pigeon hole where two or more people have entered.
Can we find 13 people living in the G.T.A. that have exactly the same number of strands of hair on their heads?

Again the answer is yes! Can you argue why?

Can we find 2 pairs of people living in the G.T.A. that have exactly the same number of strands of hair on their heads?

The pigeon hole principle is useless for solving this problem and we leave this as an unsolved mystery.