Properties of relations on a set $A$

**Reflexive:** A relation $R$ is reflexive if $(a,a) \in R$ for all $a \in A$.

**Symmetric:** A relation $R$ is symmetric if

whenever $(a_1, a_2) \in R$ then $(a_2, a_1) \in R$.

**Antisymmetric:** A relation $R$ is antisymmetric if

whenever $(a_1, a_2) \in R$ then $(a_2, a_1) \notin R$.

**NOTE:** There are relations that are neither symmetric nor antisymmetric or both symmetric and antisymmetric.

**Transitive:** A relation $R$ is transitive if

whenever $(a_1, a_2) \in R$ and $(a_2, a_3) \in R$ then $(a_1, a_3) \in R$. 

Let $A = \{1,2,3,4\}$, we can define the following relations on $A$.

$R_1 = \{(1,1), (1,2), (2,3), (1,3), (4,4)\}$

NOT reflexive: (Because $(2,2)$ is missing)

NOT symmetric: (Because the presence of $(1,2)$ requires $(2,1)$)

antisymmetric: (No occurrence of a pair, of ordered pairs, of the form $(a,b),(b,a)$)

transitive: (for every occurrence of the pair $(a,b),(b,c)$ there is $(a,c)$)
\[ R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\} \]

reflexive:\n
symmetric:

NOT antisymmetric:

transitive:
$R_3 = \{(1,3), (2,1)\}$

NOT reflexive:

NOT symmetric:

antisymmetric:

NOT transitive:
\( R_4 = \emptyset \)

NOT reflexive:

symmetric:

antisymmetric:

transitive:
\[ R_5 = A \times A = A^2 \]

reflexive:

symmetric:

NOT antisymmetric:

transitive:
Consider the relation

\[ R_6 = \{(1,1), (1,2), (2,1), (2,3), (2,2), (3,3)\} \]

NOT reflexive:

NOT symmetric:

NOT antisymmetric:

NOT transitive:
Consider the relations $<$, $\leq$, and $=$ on the Natural numbers. (less than, less than or equal to, equal to)

The relation $<$ on the Natural numbers $\{(a,b) : a,b \in \mathbb{N}, a < b\}$ is:

NOT reflexive:

NOT symmetric:

antisymmetric:

transitive:
The relation $\leq$ is on the Natural numbers $\{(a,b) : a,b \in \mathbb{N}, a \leq b\}$ is:

reflexive, NOT symmetric, antisymmetric, transitive

A relation $R$ is called a \textit{partial order} if $R$ is reflexive, antisymmetric, and transitive, so the $\leq$ relation on the natural numbers is a partial order.
The relation $=\,$ on the Natural numbers $\{(a,b) : a,b : \in \mathbb{N}, a = b\}$ is: reflexive, symmetric, transitive.

A relation $R$ is called an equivalence relation if $R$ is reflexive, symmetric, and transitive, so the $=\,$ relation on the Natural numbers is an equivalence relation.
Functions

An important special case of a relation, is a function. A relation from A to B is a *function* if every element $a \in A$ is assigned a unique element of B.

For example: A relation from A to B is *any* subset of $A \times B$, any entry in the table below can potentially be an element of a relation, and any entry can be omitted.

<table>
<thead>
<tr>
<th></th>
<th>b1</th>
<th>b2</th>
<th>b3</th>
<th>b4</th>
<th>b5</th>
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</thead>
<tbody>
<tr>
<td>a1</td>
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<tr>
<td>a4</td>
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</tbody>
</table>

However, a function would require that *exactly* one entry per row of the table is present.
Vocabulary

Suppose $f$ is a function from the set $A$ to the set $B$. Then we say that $A$ is the domain of $f$ and $B$ is the codomain of $f$. (Synonyms for codomain are: target set & range)

Notation

Let $f$ denote a function from $A$ to $B$, then we write:

\[ f : A \rightarrow B \]

which is pronounced “$f$ is a function from $A$ to $B$”, or “$f$ maps $A$ into $B$”.

If $a \in A$, and $b \in B$ we can write:

\[ f(a) = b \]

to denote that the function $f$ maps the element $a$ to $b$. 
More Vocabulary

We can say that $b$ is the image of $a$ under $f$.

More notation.

A function can be expressed by a formula (written as an equation, as illustrated by the following example:

$$f(x) = x^2 \text{ for } x \in \mathbb{R}$$

In this example $f$ is the function and $x$ is the variable.

Sometimes we can express the image of a variable (the independent variable) by a dependent variable as follows:

$$y = x^2$$
Injective(one-to-one), Surjective(onto), Bijective(one-to-one and onto) functions.

A function $f: A \rightarrow B$ is a one-to-one function if for every $a \in A$ there is a distinct image in $B$. A one-to-one function is also called an injective function or an injection.

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ and $f(x) = 2^x$.

$f(x) = 2^x$ is one-to-one because there is a distinct image for every $x \in \mathbb{R}$, that is if $2^x = 2^y$ then $x = y$. 
A function \( f: A \rightarrow B \) is an onto function if every \( b \in B \) is an image. An onto function is also called a surjective function or a surjection.

Let \( f: \mathbb{R} \rightarrow \mathbb{R} \) and \( f(x) = x^3 - x \).

\[ f(x) = x^3 - x \] is onto because the pre-image of any real number \( y \) is the solution set of the cubic polynomial equation \( x^3 - x - y = 0 \) and every cubic polynomial with real coefficients has at least one real root.

Note: \( f(x) = x^3 - x = x(x^2 - 1) \) is not one-to-one because \( f(x) = 0 \) for \( x = -1, x = +1, x = 0 \)

Note: \( f(x) = 2^x \) is not onto because \( 2^x > 0 \) for all \( x \in \mathbb{R} \).
A function that is both one-to-one and onto is called a **bijective function** or a **bijection**.

Let $f : \mathbb{R} \to \mathbb{R}$ and $f(x) = 2x$

$f(x) = 2x$ is one-to-one
because we get a distinct image for every pre-image.

$f(x) = 2x$ is onto because every $y \in \mathbb{R}$ is an image. So $f(x) = 2x$ is a bijection.
Bijective functions \textit{invertible} functions. That is suppose that \( f \) is a bijective function on the set \( A \). Then \( f^{-1} \) denotes the inverse of the function \( f \), meaning that whenever \( f(x) = y \) we have \( f^{-1}(y) = x \).

In our previous example we saw that function \( f(x) = 2x \) is a bijective function. In this case we can define \( f^{-1}(x) = x/2 \), so we get \( f^{-1}(2x) = x \).
Let $f: \mathbb{R} \to \mathbb{R}$ and $f(x) = x^2$

Observe that $f(x) = x^2$ is a function because every $x \in \mathbb{R}$ has a distinct image. However, $f(x) = x^2$ is neither one-to-one (because $f(x) = f(-x)$) or onto ($f(x) \geq 0$).
Composition of functions

**Notation:** Suppose we have functions $f: A \rightarrow B$ and $g : B \rightarrow C$, then the composition of $f$ and $g$ written as $g \circ f$ is defined as:

$$(g \circ f)(a) = g(f(a)).$$

(NOTE: carefully notice the order of $f$ and $g$ on the two sides of the equation.)

So for example let $f: \mathbb{R} \rightarrow \mathbb{R}$ be $f(x) = x^2$ and let $g: \mathbb{R} \rightarrow \mathbb{R}$ be $g(x) = 5x$. Then an example of a composition of $f$ and $g$ could be:

$$(g \circ f)(2) = g(f(2)) = g(4) = 16$$