1. Consider the following relations on the set $A = \{1, 2, 3, 4\}$: $R = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$, $\emptyset = \{\}$, $A^2 = A \times A$.

(a) (3) List the relations from above that are reflexive and justify your answer.
   $R$ is reflexive because $(a, a) \in R$ for all $a \in A$.
   $A^2$ is reflexive because $(a, a) \in A^2$ for all $a \in A$.
   $\emptyset$ is not reflexive because $(1, 1) \notin \emptyset$.

(b) (3) List the relations from above that are symmetric and justify your answer.
   $\emptyset$ is symmetric because there are no pairs of the form $(a, b)$.
   $R$ and $A^2$ are symmetric because for every pair $(a, b), a, b \in A$ we have the pair $(b, a)$.

(c) (3) List the relations from above that are antisymmetric and justify your answer.
   $R, \emptyset$ are antisymmetric because there are no pairs of the form $(a, b), a \neq b, a, b \in A$.
   $A^2$ is not antisymmetric because $\{(1, 2), (2, 1)\} \subset A^2$. 
2. (6) Consider the relation $R = \{(a, b) \in \mathbb{N}^2 : |a - b| \leq 2\}$. For example $(1, 3) \in R$ and $(3, 1) \in R$ but $(1, 4) \notin R$. Is $R$ an equivalence relation? Explain your answer.

An equivalence relation must be reflexive, symmetric and transitive. $R$ is not transitive because $|1 - 3| \leq 2$ and $|3 - 5| \leq 2$ but $|1 - 5| \notin 2$.

3. (6) Let $n \in \mathbb{N}$ and $P(n)$ be the proposition:

$$\sum_{i=1}^{n} 2^i = 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 2$$

Use mathematical induction to prove that $P(n)$ is true for all $n \in \mathbb{N}$.

**Base:** $n = 1, 2 = 2^2 - 2$.

**Induction Hypothesis:** $P(k)$ is true, that is:

$$\sum_{i=1}^{k} 2^i = 2^{k+1} - 2$$

**Induction Step:** Show that $P(k)$ true can be used to prove that $P(k + 1)$ is true.

$$\sum_{i=1}^{k+1} 2^i = \sum_{i=1}^{k} 2^i + 2^{k+1}$$

$$= 2^{k+1} - 2 + 2^{k+1}$$

$$= 2^{k+2} - 2$$
4. (6) Let $P(n)$ be the proposition

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{n \times (n + 1)} = \frac{n}{n + 1}$$

Use mathematical induction to prove that $P(n)$ is true for all $n \in \mathbb{N}$.

**Base:** $n = 1$, $\frac{1}{1 \times 2} = \frac{1}{2}$.

**Induction Hypothesis:** $P(k)$ is true, that is:

$$\sum_{i=1}^{k} \frac{1}{i(i + 1)} = \frac{k}{k + 1}$$

**Induction Step:** Show that $P(k)$ true can be used to prove that $P(k + 1)$ is true.

$$\begin{align*}
\sum_{i=1}^{k+1} \frac{1}{i(i + 1)} &= \sum_{i=1}^{k} \frac{1}{i(i + 1)} + \frac{1}{(k + 1)(k + 2)} \\
&= \frac{k}{k + 1} + \frac{1}{(k + 1)(k + 2)} \\
&= \frac{k(k + 2) + 1}{(k + 1)(k + 2)} \\
&= \frac{k^2 + 2k + 1}{(k + 1)(k + 2)} \\
&= \frac{(k + 1)(k + 2)}{(k + 1)(k + 2)} \\
&= \frac{k + 1}{k + 2}
\end{align*}$$
5. Let $\mathbb{R}^+$ denote the positive real numbers. Consider the function $f : \mathbb{R} \to \mathbb{R}^+$ such that $f(x) = 2^x$, as plotted below.

![Graph of $f(x) = 2^x$.](image)

**Figure 1: $f(x) = 2^x$**

(a) (2) Is $f$ a one-to-one function? Explain why or why not?

$f$ is one-to-one because there is a unique image for every $x \in \mathbb{R}$, that is, if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

(b) (2) Is $f$ an onto function? Explain why or why not?

$f$ is an onto function because every element in $\mathbb{R}^+$ has a pre-image in $\mathbb{R}$.

(c) (2) Is $f$ a bijective function? Explain why or why not?

$f$ is a bijective function because $f$ is both one-to-one and onto.