(1) Evaluate

(a) \(|3 - 7| = | - 4| = 4\)
(b) \(|1 - 4| - |2 - 9| = | - 3| - | - 7| = -4\)
(c) \(|-6 - 2| - |2 - 6| = | - 8| - | - 4| = 4\)

(2) Find the quotient \(q\) and remainder \(r\), as given by the Division Algorithm theorem for the following examples. Recall we want to find \(r\), \(0 \leq r < |r|\), such that \(a = qb + r\), where all values are integers.

(a) \(a = 500, \ b = 17\).
\[500 = 29 \times 17 + 7 \text{ so } r = 7.\]
(b) \(a = -500, \ b = 17\).
\[-500 = -30 \times 17 + 10 \text{ so } r = 10.\]
(c) \(a = 500, \ b = -17\).
\[500 = -29 \times -17 + 7 \text{ so } r = 7\]
(d) \(a = -500, \ b = -17\).
\[-500 = 30 \times -17 + 10 \text{ so } r = 10\]
(3) Show that \( c \mid 0 \), for all \( c \in \mathbb{Z}, c \neq 0 \).

\[
\frac{0}{c} = 0 \text{ for all } c \in \mathbb{Z}, c \neq 0. \text{ Note: } \frac{0}{0} \text{ is undefined.}
\]

(4) Let \( a, b, c \in \mathbb{Z} \) such that \( c \mid a \) and \( c \mid b \). Let \( r \) be the remainder of the division of \( b \) by \( a \), that is there is a \( q \in \mathbb{Z} \) such that \( b = qa + r, 0 \leq r < |b| \). Show that under these condition we have \( c \mid r \).

Since \( c \mid a \) and \( c \mid b \) we can write:

\[
(1) a = cp_a \text{ and } b = cp_b, \text{ such that } p_a, p_b \in \mathbb{Z}.
\]

So we can rewrite \( b = qa + r \) as:

\[
 cp_b = qcp_a + r
\]

and this simplifies to:

\[
 c(p_b - qp_a) = r
\]

Since \( p_b - qb_a \) is an integer we can conclude that \( c \mid r \).

(5) Let \( a, b \in \mathbb{Z} \) such that \( 2 \mid a \). (In other words \( a \) is even.) Show that \( 2 \mid ab \).

This is just a special case of the divisibility theorem that states if \( c \mid a \) then for any integer \( b, c \mid ab \).
(6) Let $a \in \mathbb{Z}$ show that $3|a(a+1)(a+2)$, that is the product of three consecutive integers is divisible by 3.

Observe that we can write $a = 3q + r$ where $r \in 0, 1, 2$.

Case 0: If $r = 0$ a is divisible by 3 and since $(a+1)(a+2)$ is an integer it follows that $3|a(a + 1)(a + 2)$.

Case 1: If $r = 1$, add 2 to both sides of the equation $a = 3q + 1$ to get $a + 2 = 3q + 3 = 3(q + 1)$ thus $a + 2$ is divisible by 3 and since $a(a + 1)$ is an integer it follows that $3|a(a + 1)(a + 2)$.

Case 2: If $r = 2$, add 1 to both sides of the equation $a = 3q + 1$ to get $a + 1 = 3q + 3 = 3(q + 1)$ thus $a + 1$ is divisible by 3 and since $a(a + 2)$ is an integer it follows that $3|a(a + 1)(a + 2)$.

(7) Use induction to prove $n^3 + 2n$ is divisible by 3, for all $n \in \mathbb{N}, n \geq 1$.

Base: $3|1^3 + 2$

Induction Hypothesis: Assume that $k^3 + 2k$ is divisible by 3, for $k \geq 1$.

Induction Step: Goal: Show that $3|(k + 1)^3 + 2(k + 1)$ using the induction hypothesis.

We begin by manipulating the expression $(k+1)^3+2(k+1)$ as follows:
\[(k + 1)^3 + 2(k + 1) = k^3 + 3k^2 + 3k + 1 + 2k + 2 \]
\[= k^3 + 2k + 3(k^2 + k + 1)\]

Observe that \(3|k^3 + 2k\) by the induction hypothesis and \(3|3(k^2 + k + 1)\). So \(3|k^3 + 2k + 3(k^2 + k + 1)\).

(8) Show that any integer value greater than 2 can be written as \(3a + 4b + 5c\), where \(a, b, c\) are non-negative integers, that is \(a, b, c \in \mathbb{Z}, a, b, c \geq 0\).

We use the second form of mathematical induction.

Base: \(3 = 3 \times 1 + 4 \times 0 + 5 \times 0\), and \(4 = 3 \times 0 + 4 \times 1 + 5 \times 0\), and \(5 = 3 \times 0 + 4 \times 0 + 5 \times 1\).

Induction Hypothesis: All values \(n\) such that, \(2 < n \leq k\) can be written as \(3a + 4b + 5c\), where \(a, b, c\) are non-negative integers.

Induction Step: Consider the value \(k + 1\), and the value \(k\). By the induction hypothesis \(k = 3a + 4b + 5c\) for non-zero integers \(a, b, c\). There are 3 cases to consider.

Case 1: \(a > 0\) in the expression \(k = 3a + 4b + 5c\), therefore \(k + 1 = 3(a - 1) + 4(b + 1) + 5c\).

Case 2: \(a = 0\), \(b > 0\) in the expression \(k = 3a + 4b + 5c\), therefore \(k + 1 = 4(b - 1) + 5(c + 1)\).
Case 3: $a = 0, b = 0$, and $c > 0$ in the expression $k = 3a + 4b + 5c$,
therefore $k + 1 = 3(a + 2) + 5(c - 1)$. 