CISC-102

Fall 2014
Lecture 13

Well Ordering Principle

Let $S$ be a non-empty set of positive integers. Then $S$ contains a least element $a$, such that $a \leq s$ for all $s$ in $S$.

- Observe that $S$ could be an infinite set
- Well ordering does NOT apply to subsets of $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$. It is a special property of the positive integers, a.k.a the Natural numbers.

*a.k.a = also known as
Well Ordering Principle

• The well ordering principle is a simple observation that leads to a very powerful way to construct a proof.

Template for Well Ordering Proofs

Suppose we want to prove a proposition is true for all values of $n$ in the set of Natural numbers (Or some subset of the Natural numbers.)

For example Prove that for all $n$ in $\mathbb{N}$ that the sum of the first $n$ odd numbers is $n^2$

In general we can denote such a proposition by $P(n)$. 

Template for Well Ordering Proofs

1. Let $C$ be a set of values of $n$ where $P(n)$ is false. Using set notation we can write this as:
   \[ C = \{n : n \text{ in \mathbb{N}, } P(n) \text{ is false}\} \]

2. By the well ordering principle there is a smallest element, $k$, in $C$.

3. Using the smallest element $k$, show that $C$ is actually an empty set. That is, there are no values of $n$ where $P(n)$ is false, so $P(n)$ therefore must be true for all $n$.

All of the work of the proof goes into step 3.

Template for Well Ordering Proofs

1. $C = \{n \text{ in \mathbb{N}: the sum of the first } n \text{ odd numbers is NOT } n^2.\}$
2. By the well ordering principle there is a smallest element, $k$, in $C$.
3. If $k$ is the smallest element in $C$ then the sum of the first $k-1$ odd numbers IS $(k-1)^2$ as long as $k \neq 1$. We can verify that $1 = 1^2$, so we can safely assume that there exists a positive natural number $k-1$. 


Template for Well Ordering Proofs

3. (continued) The \(k-1\)st odd number is \(2(k-1)-1=2k-3\). So we have:
\[1 + 3 + \ldots + 2k-3 = (k-1)^2.\]
The sum of the first \(k\) odd numbers is:
\[(k-1)^2+(2k-1) = k^2-2k+1+(2k-1) = k^2\]
and this implies there is no smallest \(k\) in \(C\). If there is no smallest element of \(C\) then by the well ordering principle \(C\) must be the empty set. Therefore we can conclude that the sum of the first \(n\) odd natural numbers is \(n^2\), for all \(n\) in \(N\).

Template for Well Ordering Proofs

Recap: The proof we just saw assumed that there is a set, \(C\), of counter-examples (values of \(n\) where \(P(n)\) is false). By well ordering there must be a smallest element of \(C\) if \(C\) is not empty.

If \(k\) is the smallest value such that \(P(k)\) is false and \(k \neq 1\) then \(P(k-1)\) must be true.

We then use a “clever” manipulation of an equation based on the premise that \(P(k-1)\) is true to show that \(P(k)\) must also be true. Therefore \(C\) does not have a smallest element and must be the empty set. If there are no values of \(n\) where \(P(n)\) is false, then it follows that \(P(n)\) is true for all Natural numbers \(n\).
Template for Well Ordering Proofs

Recap: The proof we just saw assumed that there is a set, C, of counter-examples (values of n where P(n) is false). By well ordering there must be a smallest element of C if C is not empty. If \( k \) is the smallest value such that P(\( k \)) is false and \( k \neq 1 \) then P(\( k-1 \)) must be true.

We then use a “clever” manipulation of an equation based on the premise that P(\( k-1 \)) is true to show that P(\( k \)) must also be true. Therefore C does not have a smallest element and must be the empty set. If there are no values of n where P(n) is false we must conclude that P(n) is true for all values of n.

Template for Well Ordering Proofs

This method of proof is in fact another way of viewing how we prove that something is true by induction.

If \( k \) is the smallest value such that P(\( k \)) is false and \( k \neq 1 \) then P(\( k-1 \)) must be true.

This is the Base step in a proof by induction.
Template for Well Ordering Proofs

This method of proof is in fact another way of viewing how we prove that something is true by induction.

If $k$ is the smallest value such that $P(k)$ is false and $k \neq 1$ then $P(k-1)$ must be true.

This is the Induction Hypothesis in a proof by induction.

Template for Well Ordering Proofs

This method of proof is in fact another way of viewing how we prove that something is true by induction.

We then use a “clever” manipulation of an equation based on the premise that $P(k-1)$ is true to show that $P(k)$ must also be true. This is the Induction Step in a proof by induction.