Propositional Functions

Let \( p(x) \) be a propositional function that is either true or false for each \( a \) in \( A \).
That is, the domain of \( p(x) \) is a set \( A \), and the range is \( \{ \text{true}, \text{false} \} \).
Observe that the set \( A \) can be partitioned into two subsets, elements with an image that is true/false. In particular we may define the truth set of \( p(x) \) as:
\[
T_p = \{ x : x \in A, p(x) \text{ is true} \} 
\]
Propositional Functions

Examples: Consider the following propositional functions defined on the positive integers.

\[ p(x): x + 2 > 7 \ ; \ T_p = \{x : x > 5\} \]
\[ p(x): x + 5 < 3 \ ; \ T_p = \emptyset \]
\[ p(x): x + 5 > 1 \ ; \ T_p = \mathbb{N} \]

Quantifiers

There are two widely used logical quantifiers

**Universal Quantifier:** \( \forall \) (for all)
e.g. \( (\forall x \in A) \ p(x) \) (for all \( x \) in \( A \) \( p(x) \) is true)
\[ T_p = \{x : x \in A, p(x)\} = A \]

**Existential Quantifier:** \( \exists \) (there exists)
e.g. \( (\exists x \in A) \ p(x) \) (There exists an \( x \) in \( A \) such that \( p(x) \) is true)
\[ T_p = \{x : x \in A, p(x)\} \neq \emptyset \]
Quantifiers

Examples:

**Theorem B.** For all x in N, \(2^x > 2\).
This statement is false, because \(2^1 = 2\). This is a counter example and it suffices to prove that Theorem B is false.

**Theorem I.** For all x in N, \(2^x \geq 2\).
This statement is true. How would you prove this?

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Quantifiers

Examples:

**Theorem A.** There exists an x in N such that \(2^x > 2\).
This statement is true, because \(2^2 > 2\). This example suffices to prove that Theorem A is true.

**Theorem H.** There exists an x in N, such that \(2^x < x\).
This statement is false. How would you prove this?
Negating Quantifiers

Examples:
For all x in \( \mathbb{N} \), \( 2^x > 2 \).
We can also write: \( p(x) \): \( 2^x > 2 \), \( x \in \mathbb{N} \).
\( (\forall x \in \mathbb{N})p(x) \)
Which we know has a value of false.
Now consider:
\( \neg(\forall x \in \mathbb{N})p(x) \)
This is the negation of the expression.

Negating Quantifiers

Now consider:
\( \neg(\forall x \in \mathbb{N})p(x) \)
DeMorgan’s Theorem:
\( \neg(\forall x \in A)p(x) \equiv (\exists x \in A)\neg p(x) \)
Therefore by DeMorgan’s Theorem we conclude that
\( \neg(\forall x \in \mathbb{N})p(x) \equiv (\exists x \in \mathbb{N})\neg p(x) \)
Negating Quantifiers

\[ \neg (\forall x \in \mathbb{N})p(x) \equiv (\exists x \in \mathbb{N})\neg p(x) \]

Recall \( p(x): 2^x > 2, x \in \mathbb{N} \)

So we have the negation of
“for all \( x \) in \( \mathbb{N} \) \( 2^x > 2 \)” is
“there exists an \( x \) in \( \mathbb{N} \) such that \( 2^x \not> 2 \).”

Negating Quantifiers

The proof that “for all \( x \) in \( \mathbb{N} \) \( 2^x > 2 \)” is false, is also a proof that “there exists an \( x \) in \( \mathbb{N} \) such that \( 2^x \not> 2 \)” is true.
Negating Quantifiers

Now consider:
\(\neg(\exists x \in \mathbb{N})p(x)\)

DeMorgan’s Theorem:
\(\neg(\exists x \in A)p(x) \equiv (\forall x \in A)\neg p(x)\)

Therefore by DeMorgan’s Theorem we conclude that
\(\neg(\exists x \in \mathbb{N})p(x) \equiv (\forall x \in \mathbb{N})\neg p(x)\)

Recall \(p(x): 2^x > 2, x \in \mathbb{N}\)

So we have the negation of
“There exists an \(x\) in \(\mathbb{N}\) such that \(2^x > 2\)” is
“For all \(x\) in \(\mathbb{N}\) \(2^x \not> 2\).”
Negating Quantifiers

The proof that “there exists an \( x \) in \( \mathbb{N} \) such that \( 2^x > 2 \)” is true, is also a proof that “for all \( x \) in \( \mathbb{N} \) \( 2^x \not\geq 2 \)” is false.

Truth Sets

In particular we may define the truth set of \( p(x) \) as: \( T_p = \{ x : x \text{ in } A, p(x) \text{ is true} \} \)

Consider the truth set \( C_p = \{ x : x \text{ in } A, \neg p(x) \text{ is true} \} \)

The set \( C_p \) is the complement of \( T_p \)
Truth Sets

Suppose we have truth sets

\[ T_p = \{ x : x \text{ in } A, \text{ p(x) is true}\} \text{ and} \]
\[ T_q = \{ x : x \text{ in } A, \text{ q(x) is true}\} \]

Then truth set for \( \text{p(x)} \land \text{q(x)} \) is

\[ T_p \cap T_q \]

Truth Sets

Suppose we have truth sets

\[ T_p = \{ x : x \text{ in } A, \text{ p(x) is true}\} \text{ and} \]
\[ T_q = \{ x : x \text{ in } A, \text{ q(x) is true}\} \]

Then truth set for \( \text{p(x)} \lor \text{q(x)} \) is

\[ T_p \cup T_q \]
Counter Example

To show that a statement $\forall x, p(x)$ is false, is equivalent to showing that $\exists x, \neg p(x)$ is true. The element that shows that $\neg p(x)$ is true is called a counter example.

Quantifiers and Propositions

Let $A = \{1, 2, 3\}$, and $x, y \in A$.
$\exists x \ \forall y, x^2 < y + 1$;

Is this statement true or false?
Quantifiers and Propositions

Let $A = \{1,2,3\}$, and $x,y \in A$.

$\exists x \forall y, x^2 < y + 1$;

Let $x = 1$, then $x^2 < y + 1$ for all $y$ in $\{1,2,3\}$. Therefore the statement is true.

Quantifiers and Propositions

Let $A = \{1,2,3\}$, and $x,y \in A$.

$\forall x \exists y, x^2 + y^2 < 12$;

Is this statement true or false?
Let $A = \{1,2,3\}$, and $x,y \in A$.

$\forall x \ \exists y, \ x^2 + y^2 < 12$;

Let $y = 1$, then $x^2 + y^2 < 12$ for all $x$ in $\{1,2,3\}$. Therefore the statement is true.

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Let $A = \{1,2,3\}$, and $x,y \in A$.

$\forall x \ \forall y, \ x^2 + y^2 < 19$;

Is this statement true or false?
Quantifiers and Propositions

Let $A = \{1,2,3\}$, and $x,y \in A$.

$\forall x \exists y, x^2 + y^2 < 19;$

The statement is true. This can be verified by trying all pairs $x$ and $y$ taken from $A$.

We can also observe that $x^2 + y^2$ reaches its maximum value when $x=y=3$ and $9+9 < 19$.

Quantifiers and Propositions

Let $A = \{1,2,3\}$, and $x,y \in A$.

$\exists x \forall y, x^2 + y^2 < 9;$

Is this statement true or false?
Let $A = \{1,2,3\}$, and $x,y \in A$.

$\exists x \forall y, x^2 + y^2 < 9$;

This is false, for all values of $x$ in $A$ and $y = 3$ we get $x^2 + 9 > 9$.

What is the negation of the following statement?

$\exists x \forall y, p(x,y)$
Quantifiers and Propositions

What is the negation of the following statement?

A. 
\( \neg( \exists x \ \forall y, p(x,y) ) \equiv \forall x \ \exists y, \neg p(x,y) \)

Quantifiers and Propositions

What is the negation of the following statement?

\( \neg( \forall x \ \forall y, p(x,y)) \equiv \exists x \ \exists y, \neg p(x,y) \)