CISC-102 Winter 2017

Quiz 2

Solutions

1. A bijection is a function that is both one-to-one and onto. Neither of the following functions are bijections. For each case explain why it is not a bijection.

(a) \( f : \mathbb{R} \mapsto \mathbb{R} \) such that \( f(x) = x^2 - 7 \).
   \( f(x) = x^2 - 7 \) is not one-to-one because \( f(x) = f(-x) \). \( f(x) = x^2 - 7 \) is not onto because any value less than -7 is not an image.

(b) \( f : \mathbb{Z} \mapsto \mathbb{Z} \) such that \( f(x) = |x| - 7 \).
   \( f(x) = |x| - 7 \) is not one-to-one because \( f(x) = f(-x) \). \( f(x) = |x| - 7 \) is not onto because any value less than -7 is not an image.

(c) \( f : \mathbb{Z} \mapsto \mathbb{Z} \) such that \( f(x) = x^3 - x \).
   \( f(x) = x^3 - x \) is not one-to-one because \( f(1) = f(0) = 0 \).

2. ( 4 ) Consider a mapping \( M : A \mapsto B \), such that \( M \) is \( A \times B \). Provide an example of non-empty sets \( A \) and \( B \) so that \( M \) is a function.
   \( B \) needs to have exactly one element, otherwise \( A \times B \) is not a function. A possible example \( A = B = \{1\} \), yielding the function \( \{(1,1)\} \).

3. Let \( R_1 \) be a relation on \( \mathbb{Z} \setminus \{0\} \) such that \( (a,b) \in R_1 \) if \( a \times b > 0 \).

   (a) ( 2 ) Is \( R_1 \) reflexive? Explain your answer.
   \( R_1 \) is reflexive because \( a^2 > 0 \) for all \( a \in \mathbb{Z} \setminus \{0\} \).

   (b) ( 2 ) Is \( R_1 \) symmetric? Explain your answer.
   \( R_1 \) is symmetric because \( a \times b = b \times a \) for all \( a, b \in \mathbb{Z} \setminus \{0\} \).

   (c) ( 2 ) Is \( R_1 \) antisymmetric? Explain your answer.
   \( R_1 \) is not anti-symmetric because \( a \times b = b \times a \) for all \( a, b \in \mathbb{Z} \setminus \{0\} \) and in particular when \( a \neq b \).

   (d) ( 2 ) Is \( R_1 \) transitive? Explain your answer.
   \( R_1 \) is transitive because \( a \times b > 0 \) and \( b \times c > 0 \) implies that \( a \times c > 0 \) for all \( a, b, c \in \mathbb{Z} \setminus \{0\} \). (In essence all three must be positive or all three must be negative.)
NOTE: $R_1$ is an equivalence relation partitioning $\mathbb{Z}\setminus\{0\}$ into two classes, positive integers, and negative integers.

4. Let $R_2$ be a relation on $\mathbb{N}$ such that $(a, b) \in R_2$ if $a|b$.

   (a) (2) Is $R_2$ reflexive? Explain your answer.
   $R_2$ is reflexive because $a|a$ for all $a \in \mathbb{N}$.

   (b) (2) Is $R_2$ symmetric? Explain your answer.
   $R_2$ is not symmetric because $1|2$ but $2 \nmid 1$.

   (c) (2) Is $R_2$ antisymmetric? Explain your answer.
   $R_2$ is anti-symmetric because if $a|b$ and $b|a$ then $a = b$ for all $a \in \mathbb{N}$.

   (d) (2) Is $R_2$ transitive? Explain your answer.
   $R_2$ is transitive because if $a|b$ and $b|c$ then $a|c$ for all $a, b, c \in \mathbb{N}$.

   NOTE: $R_2$ is a partial order.

5. (4) The Division Algorithm Theorem can be stated as:
Let $a, b \in \mathbb{Z}, b \neq 0$, then there exists $q, r \in \mathbb{Z}$, such that: $a = bq + r, 0 \leq r < |b|$. 
Show using the Division Algorithm theorem, that if $a|b$ and $a|c$ then $a|(b - c)$. 
a|b and a|c implies that there exists integers $p_b$ and $p_c$ such that $b = p_b \times a$ and $c = p_c \times a$. Therefore $b - c = a(p_b - p_c)$, and we easily conclude that $a|(b - c)$.

6. (6) Consider the following recursively defined function

$$
\begin{align*}
    f(1) & = 1 \\
    f(n) & = f(n-1) \times n \text{ for } n \in \mathbb{N}, n > 1.
\end{align*}
$$

Use mathematical induction to prove that $f(n) = n!$ for all $n \in \mathbb{N}$.

The first form of induction will suffice.

**Base:** $f(1) = 1!$

**Induction Hypothesis:** Assume that $f(k) = k!$ for some fixed value $k, k \geq 1$

**Induction Step:**

$$
\begin{align*}
    f(k + 1) & = f(k) \times (k + 1) \text{(using the definition of the function } f) \\
            & = k! \times (k + 1) \text{(using the induction hypothesis)} \\
            & = (k + 1)! \quad \square
\end{align*}
$$