Read the questions carefully. Please clearly state any assumptions that you make that are not explicitly stated in the question. Please answer all questions in the space provided. Use the back of pages for scratch work. There are 4 pages to this quiz. Note that \((x)\) denotes the question is worth \(x\) points.

**CALCULATORS ARE NOT PERMITTED.**
Note: There is no need to simplify your expressions.

1. Let \(a = 21\), and \(b = 15\).

   (a) \((4)\) Find \(g = \gcd(a, b)\). Show the steps used by Euclid’s algorithm to find \(\gcd(a, b)\).

   \[
   \begin{align*}
   21 &= (1) 15 + 6 \\
   15 &= (2) 6 + 3 \\
   6 &= (2) 3 + 0 \\
   g &= 3
   \end{align*}
   \]

   \[g = \gcd(21, 15) = \gcd(15, 6) = \gcd(6, 3) = \gcd(3, 0) = 3\]

   (b) \((2)\) Find \(l = \text{lcm}(a, b)\).

   \[\frac{|a \cdot b|}{g} = \frac{21 \cdot 15}{3} = (21)(15) \div 3\]
2. (4) In how many different ways can the letters TORONTO be rearranged?

7 letters, 3 O's, 2 T's

so there are \( \frac{7!}{3! 2!} \) different ways to rearrange TORONTO.

3. (4) Write out the 4 residue classes \((\mod 4)\) for integers in the range -10 to 10.

\[
\begin{align*}
\{0\} & = \{8, -8, 4, 0, 4, 8\} \\
\{1\} & = \{3, -3, 1, 5, 9\} \\
\{2\} & = \{2, -10, -6, -2, 2, 6, 10\} \\
\{3\} & = \{3, -9, -5, -1, 3, 7\}
\end{align*}
\]

4. (4) Prove that any set of 5 natural numbers will always have two numbers \(n_1\) and \(n_2\) such that 4|\((n_1 - n_2)\). Hint: Use the Pigeon Hole Principle.

There are 4 residue classes \((\mod 4)\) so there must be at least 2 numbers, \(n_1, n_2\), in the same residue class (by the Pigeon Hole Principle).

Therefore, \(4 \mid (n_1 - n_2)\)

5. (4) Consider a bag containing 52 balls numbered from 1 to 52. In how many ways can 5 balls be selected (without ordering)?

There are \(\binom{52}{5}\) ways to select 5 balls from 52 without ordering.
6. (4) Consider a bag containing 52 balls numbered from 1 to 52. In how many ways can 5 balls be selected (without ordering) so that the 5 numbers are consecutive (for example 2,3,4,5,6)?

Note that the lowest numbered ball defines 5 numbers in sequence. The highest the lowest numbered ball can be is 48, thus there are:

\[
\binom{48}{1} \text{ ways to select 5 consecutively numbered balls from 52.}
\]

7. At the bagel shop there are 5 varieties of bagel; sesame, poppy, plain, multigrain, and all dressed. The shop is well stocked so that there are always several dozens of each type of bagel.

(a) (4) In how many different ways can a dozen bagels be chosen? Note: a dozen = 12 bagels.

This is equivalent to counting binary strings with 12 0’s and 4 1’s; that is, \( \binom{12+4}{12} \) or

\[
\frac{(12+4)!}{12! \cdot 4!}
\]

(b) (4) In how many different ways can a dozen bagels be chosen so that at least 6 of them are sesame?

If we pre-select the sesame seed bagels we have 6 more to select. This is equivalent to counting binary strings with 6 0’s and 4 1’s; that is,

\[
\frac{(6+4)!}{6! \cdot 4!}
\]
8. Recall: The binomial theorem can be stated as:

\[(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i.\]

(2) Use the binomial theorem to expand the product \((x + y)^4\).

\[\binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} xy^3 + \binom{4}{4} y^4\]

9. (4) Show that

\[\binom{n}{0} 2^n - \binom{n}{1} 2^{n-1} + \binom{n}{2} 2^{n-2} - \binom{n}{3} 2^{n-3} + \cdots + \binom{n}{n} (-1)^n = 1\]

HINT: Use the Binomial theorem.

We can write the sum above as

\[\sum_{i=0}^{n} \binom{n}{i} 2^{n-i} (-1)^i\]

By the Binomial theorem this sum is equal to \((2-1)^n = 1\)