CISC-102

Winter 2019

Quiz 3

Solutions

April 1 ~~March 28,~~ 2019

1. (2) How many ways are there to select a 5 card poker hand such that there are two pairs and a 5th card. That is, there are two cards of the same value, another two of the same value different from the first, and a fifth card with value different from the two pairs. Recall a deck of cards has 13 different values, where each value comes in 4 different suits.

   (a) $\binom{13}{2}\binom{12}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1}$

   (b) $\binom{13}{1}\binom{12}{1}\binom{4}{2}^2\binom{11}{1}\binom{4}{1}$

   (c) $\underline{\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1}}$

   (d) none of the above

2. (2) Which of the following terms belongs to the expansion of $(x+y)^7$?

   (a) $\binom{7}{1}x^7$

   (b) $\binom{7}{2}x^3y^4$

   (c) $\binom{7}{6}x^6y^2$

   (d) <u>None of the above</u>

3. (2) Consider the sum:

$$S_n = \sum_{i=0}^{n} \binom{n}{i} 2^{n-i}(2)^i.$$

Which of one of the following is true?

(a) $S_n = 5^n$
(b) $S_n = 0$
(c) $S_n = 6^n$
(d) <u>None of the above.</u>

4. (2) Consider a bag containing 10 balls numbered from 1 to 10. In how many ways can 5 balls be selected, without ordering, and without replacement, so that all 5 numbers are even or all 5 are odd?

(a) $\binom{10}{5} + \binom{10}{5}$
(b) $\binom{5}{5}\binom{5}{5}$
(c) $\underline{\binom{5}{5} + \binom{5}{5}}$
(d) none of the above

5. (2) Which one of the following equations is true for all natural numbers k.

(a) $\binom{k+1}{k} = \binom{k+2}{k+1}$
(b) $\underline{\binom{k+1}{0} = \binom{k+2}{0}}$
(c) $\binom{k+1}{2} = \binom{k+1}{k}$
(d) none of the above

6. (2) Which of the following is true?

(a) $\binom{n}{k} \geq n^k$
(b) $\underline{\binom{n}{k} \leq n^k}$
(c) $\binom{n}{k} \leq \frac{n^k}{k^k}$
(d) none of the above

7. (2)The Fibonacci function is defined recursively as $F(1) = 1, F(2) = 1$, and $F(n) = F(n-1) + F(n-2)$ for all natural numbers $n \geq 3$. Which of the following is false?

   (a) $F(5) = 5$
   (b) $F(6) = F(5) + F(4)$
   (c) $F(7) = F(5) + 2F(4) + F(3)$
   (d) none of the above

8. (2) Which of the following expressions is false?

   (a) $p \to q \equiv q \to p$
   (b) $\neg(p \vee q) \equiv \neg p \wedge \neg q$
   (c) $p \to q \equiv \neg q \to \neg p$
   (d) none of the above

9. (2) Which of the following expressions is always true?

   (a) $\neg p \vee p$
   (b) $\neg p \wedge p$
   (c) $\neg p \wedge \neg p$
   (d) none of the above

10. (4) Consider the equation

$$x_1 + x_2 + x_3 + x_4 = 11$$

A natural number solution to this equation assigns natural numbers (integers $x, x \geq 1$) to the variables x_1, x_2, x_3, x_4 so that the sum is 11. For example one possible solution is $x_1 = 2, x_2 = 6, x_3 = 1, x_4 = 2$. How many distinct natural number solutions are there to this equation?

Solution First preassign 1 to each variable, x_1, x_2, x_3, x_4. The number of non-negative integer solutions of $x_1 + x_2 + x_3 + x_4 = (11 - 4)$ is equivalent to the number of binary strings of length 11 - 4 + 3 = 10 using 3 1's and 7 0's, which is:

$$\frac{10!}{7!3!} = \binom{10}{7} = \binom{10}{3}$$

11. (4) Complete the truth table below, adding columns as needed, for the proposition:

$$\neg(p \wedge \neg q).$$

p	q	$\neg q$	$p \wedge \neg q$	$\neg(p \wedge \neg q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

12. Consider the logical argument:

$$p, p \to q \vdash q.$$

(a) (2) Rewrite the logical argument as a logical expression.

Solution $p \wedge (p \to q) \to q$.

(b) (4) Complete the truth table below, adding columns as needed to determine whether the argument above is valid or not. After you have completed the table explain your conclusion in a sentence or two.

p	q	$p \to q$	$p \wedge (p \to q)$	$p \wedge (p \to q) \to q$	
T	T	T	T	T	
T	F	F	F	T	
F	T	T	F	T	
F	F	T	F	T	

The derived logical expression is a tautology, thus we can conclude that the given argument is valid.

13. (6) The Fibonacci function is defined recursively as $F(1) = 1, F(2) = 1$, and $F(n) = F(n-1) + F(n-2)$ for all $n \geq 3$. Prove using mathematical induction that the sum of the first n Fibonacci numbers is equal to $F(n+2) - 1$, that is:

$$\sum_{i=1}^{n} F(i) = F(n+2) - 1$$

is true for all natural numbers n.

Base: F(1) = F(3)-1 = F(2) = 1.

Induction Hypothesis: Assume that:

$$\sum_{i=1}^{k} F(i) = F(k+2) - 1$$

is true for a natural number $k \geq 1$.

Induction Step:

$$\sum_{i=1}^{k+1} F(i) = \sum_{i=1}^{k} F(i) + F(k+1) = F(k+2) - 1 + F(k+1) = F(k+3) - 1$$