Problems

1. Illustrate DeMorgan’s Law $(A \cap B)^c = A^c \cup B^c$ using Venn diagrams.

![Venn Diagrams](image)

Figure 1: $(A \cap B)$ is shown in (a), and (c) and (d) illustrate $B^c$ and $A^c$ respectively. Finally (b) shows that $(A \cap B)^c = A^c \cup B^c$

2. Let $A_i = \{1, 2, 3, \ldots, i\}$ for all $i \in \mathbb{N}$. For example $A_4 = \{1, 2, 3, 4\}$.

What are the elements of the set:

(a) What are the elements of the set $\bigcup_{i=1}^{n} A_i$ ?

$$\bigcup_{i=1}^{n} A_i = \{1, 2, \ldots, n\}$$

(b) What are the elements of the set $\bigcap_{i=1}^{n} A_i$ ?

$$\bigcap_{i=1}^{n} A_i = \{1\}$$
3. Observe that $A \subseteq B$ has the same meaning as $A \cap B = A$. Draw a Venn diagram to illustrate this fact.

See Figure 2. If $A \subseteq B$ then every element $x \in A$ is also an element in $B$, which in turn implies that $A \cap B = A$.

4. Use a Venn diagram to show that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

See Figure 2. $A \subseteq B$ implies that every element of $A$ is also in $B$, $x \in A$ implies $x \in B$. Similarly $B \subseteq C$ implies that every element of $B$ is also in $C$, $y \in B$ implies $y \in C$. Thus $A \subseteq C$.

5. Use the Principle of Exclusion and Inclusion to show that $|A \cup B| + |A \cap B| = |A| + |B|$.

It may help your understanding if you first explore an example such as $A = \{1,2,3\}$ and $B = \{3,4\}$.

By the Principle of Inclusion Exclusion we have $|A| + |B| - |A \cap B| = |A \cup B|$. These quantities are just non-negative integers so if we add $|A \cap B|$ to the right and left side of the equation, we get the desired result.

6. What are the cardinalities of the following sets?

   (a) $A = \{\text{winter, spring, summer, fall}\}$. $|A| = 4$.
   (b) $B = \{x : x \in \mathbb{Z}, 0 < x < 7\}$. $|B| = 6$.
   (c) $P(B)$, that is, the power set of $B$. $|P(B)| = 2^6 = 64$.
   (d) $C = \{x : x \in \mathbb{N}, x \text{ is even}\}$ This set has infinitely many elements.

7. Suppose that we have a sample of 100 students at Queen’s who take at least one of the following language courses, French-101, Spanish-101, German-101. Also suppose that 65 take French-101, 45 take German-101, 42 take Spanish 101, 20 take French-101 and German-101, 25 take French-101 and German-101, 25 take French-101 and Spanish-101, and 15 take German-101 and Spanish-101.

   (a) How many students take all three language courses?
Let $F$, $S$, and $G$ denote the sets of students taking French, Spanish, and German respectively. The Principle of Inclusion and Exclusion tells us that
\[ |F \cup S \cup G| = |F| + |S| + |G| - |F \cap S| - |S \cap G| - |F \cap G| + |F \cap S \cap G| \]

The problem statement gives us values for each quantity in the equation except for $|F \cap S \cap G|$. We can now simply fill in the numbers and solve for $|F \cap S \cap G|$, as follows:

\[ 100 = 65 + 42 + 45 - 25 - 15 - 20 + |F \cap S \cap G| \]

So we conclude that $|F \cap S \cap G| = 8$

(b) Draw a Venn diagram representing these 100 students and fill in the regions with the correct number.

(c) How many students take exactly 1 of these courses? Using the Venn diagram we can deduce that $28 + 10 + 18 = 56$ students take exactly one of the language courses.

(d) How many students take exactly 2 of these courses? Using the Venn diagram we can deduce that $17 + 12 + 7 = 36$ students take exactly two courses.

![Venn Diagram](image)

Figure 3: Language Courses Venn Diagram

8. Let $S = \{a,b,c,d,e,f,g\}$. Determine which of the following are partitions of $S$:

(a) $P_1 = \{a,c,e\}, \{b\}, \{d,g\}$ No, because $f$ is missing from the union of the sets.

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(b) \( P2 = \{ \{a, b, e, g\}, \{c\}, \{d, f\} \} \) Yes. The union of the sets is \( S \), and the pairwise intersections of the sets are empty.

(c) \( P3 = \{ \{a, e, g\}, \{c, d\}, \{b, e, f\} \} \) No, because the intersection of \( \{a, e, g\} \cap \{b, e, f\} \) is not empty.

(d) \( P4 = \{ \{a, b, c, d, e, f, g\} \} \) Yes, this is technically a partition, but a very uninteresting one.

9. Recall that the union operation is associative, that is \( A \cup (B \cup C) = (A \cup B) \cup C \). Show that the relative complement set operation is not associative, that is, \( A \setminus (B \setminus C) = (A \setminus B) \setminus C \), is incorrect for some sets \( A, B, C \). (Note if relative complement is associative then the equation must be true for all sets \( A, B, C \).)

Let \( A = \{1, 2, 3\} \ B = \{1, 2\} \) and \( C = \{2, 3\} \). \( A \setminus (B \setminus C) = \{2, 3\} \) and \( (A \setminus B) \setminus C = \emptyset \)

10. Consider a set \( S \) of \( n \) elements, such that \( \{a, b\} \subseteq S \).

(a) What is the cardinality of the power set of \( S \setminus \{a\} \)?

We know that \( S \) has \( n \) elements so \( S \setminus \{a\} \) has \( n - 1 \) elements. The power set of \( S \setminus \{a\} \) has \( 2^{n-1} \) elements.

(b) What is the cardinality of the power set of \( S \setminus \{a, b\} \)?

The power set of \( S \setminus \{a, b\} \) has \( 2^{n-2} \) elements.

(c) How many subsets of \( S \) are there that contain the element \( a \)?

Here is a way to construct the subsets of \( S \) that contain the element \( a \). For each subset \( s \in P(S \setminus \{a\}) \) construct the set \( \{a\} \cup s \). This yields all subsets of \( S \) that contain \( a \). Since there are \( 2^{n-1} \) subsets of \( S \setminus \{a\} \), there are \( 2^{n-1} \) subsets of \( S \) that contain the element \( a \).

We can obtain the same result by using a different argument. We know that there are \( 2^n \) subsets of \( S \). The subsets of \( S \setminus \{a\} \) are also subsets of \( S \) that do not contain \( a \). The total number of subsets of \( S \) is \( 2^n \). So the number of subsets of \( S \) that contain \( a \) is equal to \( 2^n - 2^{n-1} = 2^{n-1} \).

(d) How many subsets of \( S \) are there that contain the element \( a \) and exclude the element \( b \)?

Here is a way to construct the subsets of \( S \) that contain \( a \) and exclude \( b \). For each subset \( s \in P(S \setminus \{a, b\}) \) construct the set \( \{a\} \cup s \). This yields \( 2^{n-2} \) subsets of \( S \).