CISC-102 WINTER 2020

HOMEWORK 4 SOLUTIONS

(1) Determine whether the mappings given below where $f : \mathbb{R} \mapsto \mathbb{R}$ are or are not functions, and explain your decision.

(a) $f(x) = 1/x$
$f(x) = 1/x$ is not a function from $\mathbb{R}$ to $\mathbb{R}$ because $1/x$ is not defined for $x = 0$. $f(x) = 1/x$ is a functions from $\mathbb{R} \setminus \{0\}$ to $\mathbb{R}$.

(b) $f(x) = \sqrt{x}$
$f(x) = \sqrt{x}$ is not a function from $\mathbb{R}$ to $\mathbb{R}$ because $\sqrt{x}$ is not a real number for $x < 0$. Furthermore, $\sqrt{x}$ has a positive and negative value for $x \in \mathbb{R}, x > 0$. We could salvage this by defining the set $\mathbb{R}^+ = \{x : x \in \mathbb{R}, x \geq 0\}$, and consider a function from $\mathbb{R}^+$ to $\mathbb{R}^+$ defined as $f(x) = +\sqrt{x}$.

(c) $f(x) = 3x - 3$
Consider the equation:

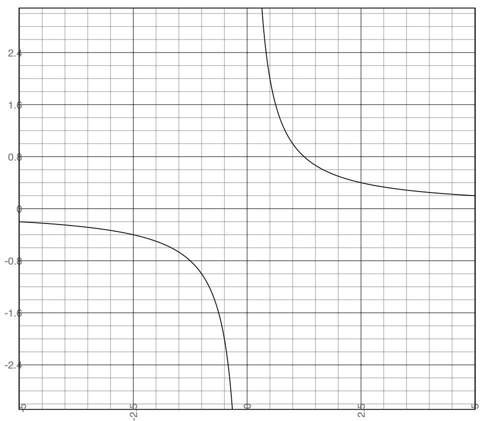
$$y = 3x - 3.$$

Observe that $3x - 3$ has a distinct image $y \in \mathbb{R}$. Therefore, $f(x) = 3x - 3$ is a function.

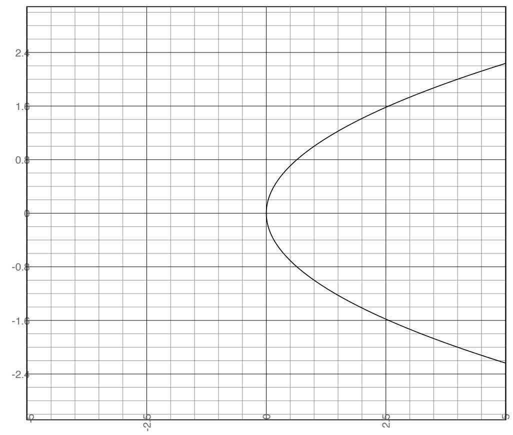
(2) Determine whether each of the following functions from $\mathbb{R}$ to $\mathbb{R}$ is a bijection, and explain your decision. HINT: Plotting these functions may help you with your decision.

(a) $f(x) = 3x + 4$
$f(x) = 3x+4$ is an onto function. Consider the equation $y = 3x+4$. For every real valued y we can find a real valued x, that is $x = y/3 - 4$. $f(x) = 3x + 4$ is a one-to-one function because, if $3x_1 + 4 = 3x_2 + 4$ then $x_1 = x_2$. Therefore we can conclude that $f(x) = 3x + 4$ is a bijection. Furthermore we have $f^{-1}(x) = x/3 - 4$

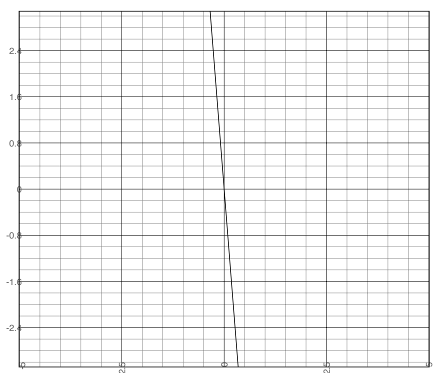
(b) $f(x) = -x^2 + 2$
$f(x) = -x^2 + 2$ is not a bijection. It is neither onto ($f(x) \leq 2$) nor one-to-one ($f(x) = 0$ for $x = +\sqrt{2}$ and for $x = -\sqrt{2}$).



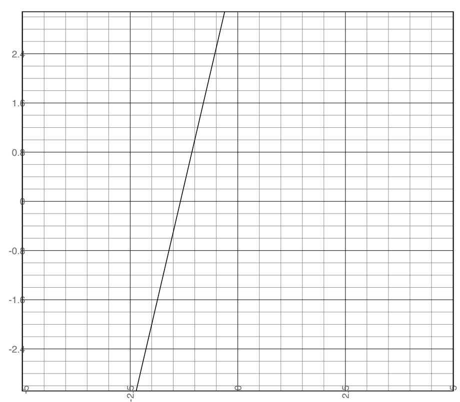
(a)



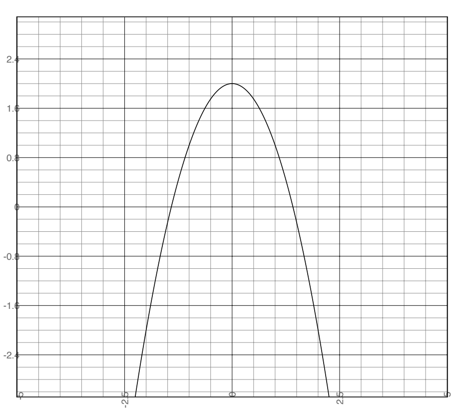
(b)



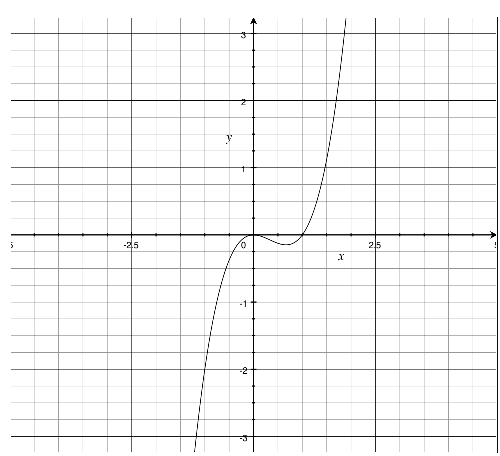
(c)



(d)



(e)



(f)

FIGURE 1. Graphs of functions for questions 1 and 2. (a) $1/x$ (b) $\sqrt{x}$ (c)$3x - 3$ (d) $3x + 4$ (e) $-x^2 + 2$ (f) $x^3 - x^2$

(c) $f(x) = x^3 - x^2$
$f(x) = x^3 - x^2$ is not a bijection. It is not one-to-one because $x^3 - x^2 = x^2(x-1)$ and is equal to 0 for $x = 1$, and $x = 0$.

(3) Suppose the function $f : A \to B$ is a bijection. What can you say about the values $|A|$ and $|B|$?

We can say that $|A| = |B|$.

(4) Consider the recursive function $T(1) = 1, T(n) = T(n-1) + 1$, for all $n \geq 2$.

(a) Use the recursive definition to obtain values T(2), T(3), and T(4).
T(2) = 2, T(3) = 3, T(4) = 4.

(b) Using the values that you obtained for T(2), T(3), and T(4), to guess the value of T(n), and then prove that it is correct using induction.

We guess that T(n) = n, and prove this using mathematical induction.

Let P(n) denote the proposition that the recursive function T(n) as defined above has the closed form solution T(n) = n.

P(n) is true for all $n \in \mathbb{N}$.

Proof: We use mathematical induction.

Base: T(1) = 1 by definition.

Induction Hypothesis: Assume that P(k) is true for some $k, k \geq 1$, that is, T(k) = k.

Induction Step: P($k+1$) is the proposition that T($k+1$) = $k+1$, and we show that it is true using the induction hypothesis.

$$\begin{aligned} T(k+1) &= T(k) + 1(\text{defintion of T(k + 1)}) \\ &= k + 1(\text{induction hypothesis}) \end{aligned}$$

We have shown that P(k) true implies that P($k+1$) is true so by the principle of mathematical induction we conclude that P(n) is true for all $n \in \mathbb{N}$. □

(5) Consider the recursive function $F(1) = 3, F(n) = 3F(n-1)$, for all $n \geq 2$.

(a) Use the recursive definition to obtain values F(2), F(3), and F(4).
F(2) = 9, F(3) = 27, F(4) = 81.

(b) Use the values that you obtained for F(2), F(3), and F(4), to guess the value of F(n), and then prove that it is correct using induction.

We guess that F(n) = 3^n, and prove this using mathematical induction.

Let P(n) denote the proposition that the recursive function F(n) as defined above has the closed form solution F(n) = 3^n.

P(n) is true for all $n \in \mathbb{N}$.

Proof: We use mathematical induction.

Base: F(1) = 3 = 3^1 by definition.

Induction Hypothesis: Assume that P(k) is true for some $k, k \geq 1$, that is, F(k) = 3^k.

Induction Step: P($k+1$) is the proposition that F($k+1$) = 3^{k+1}, and we show that it is true using the induction hypothesis.

$$\begin{aligned} F(k+1) &= 3F(k)(\text{defintion of F(k+1)}) \\ &= 3(3^k)(\text{induction hypothesis}) \\ &= 3^{k+1} \end{aligned}$$

We have shown that P(k) true implies that P($k+1$) is true so by the principle of mathematical induction we conclude that P(n) is true for all $n \in \mathbb{N}$. ☐

(6) Consider the recursive function $G(1) = x, G(n) = xG(n-1)$, for all $n \geq 2$

(a) Use the recursive definition to obtain values $G(2), G(3)$, and $G(4)$.
$G(2) = x^2, G(3) = x^3, G(4) = x^4$.

(b) Use the values that you obtained for $G(2), G(3)$, and $G(4)$, to guess the value of $G(n)$, and then prove that it is correct using induction.

We guess that G(n) = x^n, and prove this using mathematical induction.

Let P(n) denote the proposition that the recursive function G(n) as defined above has the closed form solution G(n) = x^n.

P(n) is true for all $n \in \mathbb{N}$.

Proof: We use mathematical induction.

Base: G(1) = x = x^1 by definition.

Induction Hypothesis: Assume that P(k) is true for some $k, k \geq 1$, that is, G(k) = x^k.

Induction Step: P($k+1$) is the proposition that G($k+1$) = x^{k+1}, and we show that it is true using the induction hypothesis.

$$\begin{aligned} G(k+1) &= xG(k)(\text{defintion of G(k+1)}) \\ &= x(x^k)(\text{induction hypothesis}) \\ &= x^{k+1} \end{aligned}$$

We have shown that $\mathrm{P}(k)$ true implies that $\mathrm{P}(k+1)$ is true so by the principle of mathematical induction we conclude that $\mathrm{P}(n)$ is true for all $n \in \mathbb{N}$. $\square$