CISC-102 FALL 2014

HOMEWORK 4 SOLUTIONS

Problems

(1) Explain why each of the following binary relations on the set \( S = \{1, 2, 3\} \) is not an equivalence relation on \( S \).

Recall that a relation is an equivalence relation if it is symmetric, reflexive, and transitive.

(a) \( R = \{(1, 1), (1, 2), (3, 2), (3, 3), (2, 3), (2, 1)\} \)

This relation is not reflexive, and not transitive. It is symmetric.

(b) \( R = \{(1, 1), (2, 2), (3, 3), (2, 1), (1, 2), (3, 2), (2, 3), (3, 3), (3, 1), (1, 3)\} \)

This relation is symmetric, reflexive and transitive. It is an equivalence relation.

(c) \( R = \{(1, 1), (2, 2), (3, 3), (2, 1), (1, 2), (1, 3)\} \)

This relation is not symmetric, and not transitive. It is reflexive.

(2) The sets \( \{1\}, \{2\}, \{3\} \) are the equivalence classes for a well known equivalence relation on the set \( \{1, 2, 3\} \). What is the common name and symbol for this equivalence relation?

This is ", equality.

(3) Prove using mathematical induction that every odd number \( x \in \mathbb{N} \) can be written as \( 2k - 1 \) where \( k \in \mathbb{N} \).

Base: \( 1 = 2*1-1=1 \)

Induction Hypothesis: Assume that the \( j \)th odd number can be written as \( 2j - 1 \).

Induction Step: The \( j + 1 \)st odd number is equal to the \( j \)th odd number plus 2. That is, \( 2j - 1 + 2 = 2(j + 1) - 1 \).

(4) Consider the set, \( \mathbb{Z}_0 \), of non-zero integers \( \mathbb{Z}_0 = \mathbb{Z} \setminus \{0\} = \mathbb{Z} - \{0\} \), and define the relation \( \sim \) on \( \mathbb{Z}_0 \) such that \( a \sim b \) if \( a + b \) is even.

(a) Show that \( \sim \) defines an equivalence relation on \( \mathbb{Z}_0 \).

\( \sim \) is reflexive because \( a + a = 2a \) for all non-zero integers, and \( a + a \) is always even. \( \sim \) is symmetric because if \( a + b \) is even then \( b + a \) is also even. Suppose \( a + b \) is even. Observe that both \( a \) and \( b \) must be odd or both \( a \) and \( b \) must be even. This can be demonstrated as follows:

Both even: \( a = 2i, b = 2j \), where \( i \) and \( j \) are integers, then \( a + b = 2i + 2j = 2(i + j) \) and is even.

Both odd: \( a = 2i - 1, b = 2j - 1 \), where \( i \) and \( j \) are integers, then \( a + b = 2i - 1 + 2j - 1 = 2i + 2j - 2 = 2(i + j - 1) \) and is even.
One of each: Without loss of generality let $a = 2i$, $b = 2j - 1$, where $i$ and $j$ are integers, then $a + b = 2i + 2j - 1 = 2(i + j) - 1$ and is odd. Therefore if $a + b$ is even and $b + c$ is even then $a, b, c$ are all either even or odd, and $a + c$ is even.

(b) What is the equivalence class [5]? What is the equivalence class [-5]?
   The equivalence class [5] = [-5] and is the set of odd numbers.

(c) What is the partition of $\mathbb{Z}_0$ determined by this equivalence relation.
   The partition is two equivalence classes, even numbers and odd numbers.

(5) Consider the following relations on the set $A = \{1, 2, 3\}$: $R = \{(1, 1), (1, 2), (1, 3), (3, 3)\}$, $S = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$, $T = \{(1, 1), (1, 2), (2, 2), (2, 3)\}$, $A \times A$.
   Which of the relations above are antisymmetric?
   Recall a relation, $R$, is antisymmetric if $(a, b) \in R$ implies $(b, a) \notin R$ whenever $a \neq b$.
   $R$ is antisymmetric, $S$ is not antisymmetric, $T$ is antisymmetric, $A \times A$ is not antisymmetric.

(6) Let $R$ be a relation on the set of Natural numbers such that $(a, b) \in R$ if $a \geq b$.
   Show that the relation $R$ on $\mathbb{N}$ is a partial order.
   Recall that a relation is a partial order if it is reflexive, antisymmetric, and transitive.
   $R$ is reflexive because $a \geq a$ for all $a \in \mathbb{N}$.
   $R$ is antisymmetric because $a \geq b$ implies $b \not\geq a$ if $a \neq b$.
   $R$ is transitive because if $a \geq b$ and $b \geq c$ then $a \geq c$. 