CISC-102 WINTER 2020

HOMEWORK 7
SOLUTIONS

(1) Find all Natural numbers between 1 and 50 that are congruent to 4 (mod 11).

Observe if x is a natural number such that $x \equiv 4(\mod 11)$ then $11|(x-4)$, or equivalently the difference $x-4$ is a multiple of 11. So our list is 4, 15, 26, 37, 48.

A succint way to describe all natural numbers x such that $x \equiv 4(\mod 11)$ is $[4]_{11}$.

(2) Find two Natural numbers a and b such that $2a \equiv 2b(\mod 6)$, but $a \not\equiv b(\mod 6)$.

A systematic and easy way to solve this is to consider equivalence classes mod 6. In particular $[0]_6 = 6, 12, \ldots$. We have $6 = 2 \times 3$, and $12 = 2 \times 6$, which immediately gives us our solution $a = 3$ and $b = 6$.

(3) Prove that if $a \equiv b(\mod m)$ and $c \equiv d(\mod m)$ then $a - c \equiv b - d(\mod m)$.

Observe that we need to show that $m|((a-c)-(b-d))$. Thus, $a \equiv b(\mod m)$ and $c \equiv d(\mod m)$ respectivley imply that $m|(a-b)$ and $m|(c-d)$, and $m|((a-b)-(c-d))$, and this can be rewritten as: $m|(a-c)-(b-d)$.

(4) Write out each of the 5 residue classes (mod 5) for integers in the range -10 to 10.

$[0]_5 = \{-10, -5, 0, 5, 10\}$
$[1]_5 = \{-9, -4, 1, 6\}$
$[2]_5 = \{-8, -3, 2, 7\}$
$[3]_5 = \{-7, -2, 3, 8\}$
$[4]_5 = \{-6, -1, 4, 9\}$

(5) Let S be a finite subset of the positive integers. What is the smallest value for $|S|$ that guarantees that at least two elements $x, y \in S$ have the same remainder when divided by 100. HINT: Use the pigeon hole principle.

There are 100 residue classes (mod 100) so by the pigeon hole principle any subset of the positive integers that has at least 101 elements has two or more elements that are in the same residue class, or equivalently have the same remainder when divided by 100.

(6) Prove that any set of 5 natural numbers will always have two numbers n_1 and n_2 such that $4|(n_1 - n_2)$. Hint: Use the Pigeon Hole Principle.

There are exactly 4 residue classes mod 4. By the pigeon whole principle any 5 natural numbers will have n_1 and n_2 such that n_1 and n_2 are in the same residue class (mod 4). If n_1 and n_2 are in the same residue class (mod 4), we can conclude that $4|(n_1 - n_2)$.

(7) Let T be a set of n integers. Prove that there is a subset of T whose elements sum to a value that is divisible by n.

Denote the elements of T as $t_1, t_2, \ldots, t_n$. Now consider n sums

$$s_k = \sum_{i=1}^{k} t_i.$$

We can partition the sums into residue classes ($\mod n$). If there is at least one sum s in the residue class $[0]_n$ we have $n|s$. Otherwise with n sums and at most $n-1$ residue classes by the pigeonhole principle there must be two sums s_a and s_b such that $s_a \equiv s_b (\mod n)$. Assuming $a > b$, observe that

$$s_a - s_b = \sum_{i=1}^{a} t_i - \sum_{i=1}^{b} t_i = t_{b+1} + t_{b+2} + \ldots + t_a.$$

Now it follows that $n|(s_a - s_b)$ and $s_a - s_b$ is the sum of a subset of T with a value that is divisible by n.

(8) New parents wish to give their new baby one, two, or three different names. They have a book containing 500 names that they will choose from. How many different ways can this baby be named?

If the baby is given one name then there are obviously 500 choices.
If the baby is given two names then we use the product rule to deduce that there are 500×499 choices.
If the baby is given three names then we use the product rule to deduce that there are $500 \times 499 \times 498$ choices.

We now use the sum rule to add up the choices to get a total of $500 + 500 \times 499 + 500 \times 499 \times 498$ choices.

(9) You have chosen a password that consists of 4 upper case letters from a 26 letter alphabet. How many passwords does a hacker have to try to be sure that they can break in? What if you may use both upper and lower case for your four symbol password? (Note: You may use upper and lower case letters, but that does not preclude the possibility that all of the letters are upper case or lower case.) Finally consider a password 7 symbols long and you may use both upper and lower case letters, and at least one digit $(0 \ldots 9)$.

There are 26^4 different 4 symbol passwords using only upper case letters.
There are 52^4 different 4 symbol passwords using both upper and lower case letters.
There are 62^7 different 7 symbol passwords upper and lower case letters and digits.
But this also counts passwords with no digits. Therefore we subtract the number of passwords using upper and lower case letters, but no digits, that is, $62^7 - 52^7$.

(10) How many different strings can you make using the letters TIMBITS?

The are $\frac{7!}{2!^2}$ different strings.