CISC-102
Winter 2020
Week 9

When we expand the expression:
$(x + y)^3$

we get:
$(x+y)(x+y)(x+y) = x^3 + 3x^2y + 3xy^2 + y^3$

this can also be written as follows:

$$(x+y)(x+y)(x+y) = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$$

We can reason that when we expand $(x + y)^3$, there is one way to choose a triple that is exclusively x's (with 0 y's), 3 ways to choose a triple that has 2 x's (and 1 y) , and 3 ways to choose a triple that has 1 x (and 2 y's). Finally there is 1 way to choose a triple with no x (and 3 y's).

Binomial Theorem:

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}x^0 y^n$$
$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

For all natural numbers *n*.

Proof: In the expansion of the product:

$$(x + y)\,(x + y) \cdots (x+y),$$

there $\binom{n}{k}$ ways to choose an *n*-tuple with n-*k* *x*'s and (*k* *y*'s). □

A special case of the binomial theorem should look familiar.

$$\begin{aligned}(1+1)^n &= \binom{n}{0}1^n1^0 + \binom{n}{1}1^{n-1}1 + \binom{n}{2}1^{n-2}1^2 + \cdots + \binom{n}{n}1^01^n \\ &= \sum_{k=0}^{n}\binom{n}{k}\end{aligned}$$

This is just the sum the sizes of all subsets of a set of size *n*.

Using counting to prove theorems.

Counting arguments can be useful tool for proving theorems. In each case there is also an algebraic way of proving the result. However, there is an inherent beauty in the elegant simplicity of some of these counting arguments so it's well worth looking at some examples. These proofs lack the formality of algebraic proofs. The lack of formality may make these arguments harder to grasp for some, and easier to understand for others.

The proofs we see will be to prove the validity of equations. We will count the left and right hand side of each equation and show that they count the same thing.

Binomial Coefficients

We prove identities involving binomial coefficients using counting arguments.

Theorem:

$$\binom{n}{k} = \binom{n}{n-k}$$

Proof: On the left we have the quantity $\binom{n}{k}$ which represents the number of ways to select a *k* element subset from an *n* element set, *S*. Using the analogy of selecting balls from a bag, we see that we also implicitly select the complementary subset that stays in the bag, and the number of ways to do this is as given on the right hand side of the equation is $\binom{n}{n-k}$. $\square$

Theorem A:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

Proof: On the left the quantity $\binom{n+1}{k}$ represents the number of ways to select a k element subset from an $n+1$ element set. To see what the right hand side counts we suppose that there is a "favourite" or "distinguished" element of the set, call it x.
The number of ways to select a k element subset from $n+1$ distinct objects that is guaranteed to *include* x is to pull x out and then choose the remaining $k-1$ elements in $\binom{n}{k-1}$ ways. On the other hand the number of ways to select a k element subset from $n+1$ distinct objects that is guaranteed to *exclude* x is to pull x out and then choose all k elements in $\binom{n}{k}$ ways.

Therefore the left and right hand side both count the same thing thus justifying the equation. □

And here's an alternate algebraic proof.

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

Proof:

$$\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(n-k+1)!(k-1)!} + \frac{n!}{(n-k)!(k)!} \\
&= \frac{n!k + n!(n-k+1)}{(n+1-k)!k!} \\
&= \frac{n!(k+n-k+1)}{(n+1-k)!k!} \\
&= \frac{n!(n+1)}{(n+1-k)!k!} \\
&= \frac{(n+1)!}{(n+1-k)!k!} \\
&= \binom{n+1}{k}
\end{aligned}$$

Theorem:

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

Proof: On the left the sum counts all the subsets of a set of size n. We already know that the number of subsets of a set of size n, is 2^n.

Therefore the left and right hand side both count the same thing thus justifying the equation. □

Pascal's Triangle

An easy way to calculate a table of binomial coefficients was recognized centuries ago by mathematicians in India, China, Iran and Europe.
In the west the technique is named after the French mathematician Blaise Pascal (1623-1662). In the example below each row represents the binomial coefficients as used in the binomial theorem.

$$
\begin{array}{ccccccccccccc}
 & & & & & & \binom{0}{0} & & & & & & \\
 & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
 & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
 & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & \\
 & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & \\
\binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6}
\end{array}
$$

To obtain the entries by hand in a simple way we can use the identity:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

$$\begin{array}{ccccccccccccc}
 & & & & & & \binom{0}{0} & & & & & & \\
 & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
 & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
 & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & \\
 & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & \\
\binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6}
\end{array}$$

$$\begin{array}{ccccccccccccccccc}
 & & & & & & & & 1 & & & & & & & & \\
 & & & & & & & 1 & & 1 & & & & & & & \\
 & & & & & & 1 & & 2 & & 1 & & & & & & \\
 & & & & & 1 & & 3 & & 3 & & 1 & & & & & \\
 & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & \\
 & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & \\
 & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \\
 & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & \\
1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1
\end{array}$$

Consider the sum of elements in a row of Pascal's triangle. If we label the top row 0, then it appears that row i sums to the value 2^i. Can you explain why this is the case?

$$
\begin{array}{ccccccccccccc}
 & & & & & & \binom{0}{0} & & & & & & \\
 & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
 & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
 & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & \\
 & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & \\
\binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6}
\end{array}
$$

$$
\begin{array}{ccccccccccccccccc}
 & & & & & & & & 1 & & & & & & & & \\
 & & & & & & & 1 & & 1 & & & & & & & \\
 & & & & & & 1 & & 2 & & 1 & & & & & & \\
 & & & & & 1 & & 3 & & 3 & & 1 & & & & & \\
 & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & \\
 & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & \\
 & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \\
 & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & \\
1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1
\end{array}
$$

Now let's compute the sum of squares of the entries of each row in Pascal's triangle.

$1^2 = 1$
$1^2 + 1^2 = 2$
$1^2 + 2^2 + 1^2 = 6$
$1^2 + 3^2 + 3^2 + 1^2 = 20$
$1^2 + 4^2 + 6^2 + 4^2 + 1^2 = 70$

These sums all appear in the middle row of Pascal's triangle.

$$
\begin{array}{ccccccccccccccccc}
&&&&&&&&1&&&&&&&&\\
&&&&&&&1&&1&&&&&&&\\
&&&&&&1&&2&&1&&&&&&\\
&&&&&1&&3&&3&&1&&&&&\\
&&&&1&&4&&6&&4&&1&&&&\\
&&&1&&5&&10&&10&&5&&1&&&\\
&&1&&6&&15&&20&&15&&6&&1&&\\
&1&&7&&21&&35&&35&&21&&7&&1&\\
1&&8&&28&&56&&70&&56&&28&&8&&1
\end{array}
$$

Which leads us to conjecture that:

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$$

Before proving the theorem there are two preliminary lemmas.

Lemma 1:

$$\binom{n}{k}\binom{n}{n-k} = \binom{n}{k}^2$$

For all non-negative integers n,k, $n > k$.

Proof: Since we already showed that $\binom{n}{k} = \binom{n}{n-k}$ this should be obvious. □

$$\sum_{i=0}^{k} \binom{m}{k-i}\binom{n}{i} = \binom{m+n}{k}$$

Lemma 2:

For all non-negative integers *m,n,k* such that $n \geq m \geq k$.

Proof: We use a counting argument. The right hand side can be viewed as the number of subsets of size *k* chosen from the union of two <u>disjoint</u> sets, *S* of size *m*, and *T* of size *n*. On the left we sum the choices where all *k* are from *S*, then *k-1* from *S* and 1 from *T* and so on up to all *k* chosen from set *T*. □

For example: Suppose
S = {a,b} with |S| = *m* = 2, and
T = {c,d,e} with |T| = *n* = 3 and
k = 2. So the sum on the right would be:

$$\sum_{i=0}^{2} \binom{2}{2-i}\binom{3}{i} = \binom{2}{2}\binom{3}{0} + \binom{2}{1}\binom{3}{1} + \binom{2}{0}\binom{3}{2}$$

Theorem:

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$$

for all natural numbers n ≥ 1.

Proof: Using lemma 1 we can write $\binom{n}{i}^2 = \binom{n}{i}\binom{n}{n-i}$.

Now we observe that the sum is just a special case of lemma 2, where *m* = *n*, and *k* = *n*, as follows:

$\square$

$$\sum_{i=0}^{n} \binom{n}{n-i}\binom{n}{i} = \binom{n+n}{n}$$